13th September, 2021

MAT3500/MAT4500 Topology

Mandatory assignment 1 of 1

Submission deadline

Thursday 21st October 2021, 14:30 in Canvas (canvas.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX), but use of LaTeX is strongly encouraged. The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

The assignment consists of seven parts, 1(a) through 3(a), each worth 6 points. All answers should be justified. A total score of 40 %, or 17 points, is sufficient to pass.

Problem 1. Let

$$X = \{(a, b, c) \in \mathbb{C}^3 \mid a \neq b \neq c \neq a\}$$

be the set of pairwise distinct triples of points in $\mathbb{C} \cong \mathbb{R}^2$. Each triple (a, b, c) determines a (possibly degenerate) triangle $\triangle abc$ in the plane. Let

$$Y = \mathbb{C} \times (\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0, 1\})$$

be the set of triples $(x, y, z) \in \mathbb{C}^3$ with $y \neq 0$ and $z \notin \{0, 1\}$. Give $\mathbb{C}^3 \cong \mathbb{R}^6$ the usual metric topology, and give X and Y the subspace topologies.

(a) Prove that

$$f(a, b, c) = (a, b - a, \frac{c - a}{b - a})$$

defines a homeomorphism $f \colon X \to Y$, by explaining why f is continuous and admits a continuous inverse $g \colon Y \to X$.

Let

$$A = \{(a, b, c) \in X \mid |b - a| \neq |c - b| \neq |a - c| \neq |b - a|\}$$

be the subspace of X corresponding to the non-isosceles triangles $\triangle abc$, and let $B = f(A) \subset Y$.

(b) Show that

$$B = \mathbb{C} \times (\mathbb{C} - \{0\}) \times Z$$

for a subspace $Z \subset \mathbb{C} - \{0, 1\}$.

(c) How many connected components does A have?

Problem 2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f \colon X \to Y$ be a map (= continuous function), with image $Z = f(X) \subset Y$. Factor $f = i \circ p$ with $p \colon X \to Z$ surjective and $i \colon Z \to Y$ the inclusion. Let $\mathcal{T}_1$ denote the quotient topology on Z induced by p, and let $\mathcal{T}_2$ denote the subspace topology on Z inherited from Y.

$$\begin{array}{ccc} (X, \mathcal{T}_X) & \xrightarrow{f} & (Y, \mathcal{T}_Y) \\ {\scriptstyle p}\downarrow & & \uparrow{\scriptstyle i} \\ (Z, \mathcal{T}_1) & \xrightarrow{j} & (Z, \mathcal{T}_2) \end{array}$$

(a) Show that the topology $\mathcal{T}_1$ is finer than (or equal to) the topology $\mathcal{T}_2$.

(b) Give an example of a map $f\colon X \to Y$ such that Z has two elements and $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_2$.

(c) Suppose that $(X, \mathcal{T}_X)$ is compact and $(Y, \mathcal{T}_Y)$ is Hausdorff. Prove that in this case the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal.

Problem 3. Let (X, D) be an anti-metric space, meaning that X is a set and $D\colon X \times X \to \mathbb{R}$ is a function such that

(1) $D(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $D(x, y) = D(y, x)$ for all $x, y \in X$.

(3) (opposite triangle inequality) $D(x, y) + D(y, z) \leq D(x, z)$ for all $x, y, z \in X$.

(a) Prove that X cannot have a prime number of elements.