SAMPLE SOLUTIONS FOR MANDATORY ASSIGNMENT MAT3500/MAT4500, AUTUMN 2021

1(a): Since $Y \subset \mathbb{C}^3$ is a subspace, and $\mathbb{C}^3 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ is a product space, $f\colon X \to Y$ is continuous if and only if each of the composites

$$f_i\colon X \xrightarrow{f} Y \subset \mathbb{C}^3 \xrightarrow{\pi_i} \mathbb{C}$$

are continuous, for $i \in \{1, 2, 3\}$. Here

$$f_1(a, b, c) = a \ , \quad f_2(a, b, c) = b - a \quad \text{and} \quad f_3(a, b, c) = (c - a)/(b - a)$$

are restrictions of rational functions, which are known to be continuous on the subspaces of $\mathbb{C}^3$ where they are defined, which contain X. Hence each f_i is continuous.

Solving $f(a, b, c) = (a, b - a, (c - a)/(b - a)) = (x, y, z)$ for (a, b, c) gives $a = x$, $b = x + y$ and $c = x + yz$, so we define $g\colon Y \to X$ by

$$g(x, y, z) = (x, x + y, x + yz)\,.$$

For $(x, y, z) \in Y$ we have $y \neq 0$ and $z \notin \{0, 1\}$, which implies $x \neq x + y$, $x + y \neq x + yz$ and $x + yz \neq x$. Hence g is well-defined as a function to X. Moreover,

$$g(f(a, b, x)) = g(a, b - a, (c - a)/(b - a)) = (a, a + (b - a), a + (b - a)(c - a)/(b - a)) = (a, b, c)$$
$$f(g(x, y, z)) = f(x, x + y, x + yz) = (x, x + y - x, (x + yz - x)/(x + y - x)) = (x, y, z)\,,$$

so $gf = 1_X$ and $fg = 1_Y$. Hence g is inverse to f.

Since $X \subset \mathbb{C}^3$ is a subspace, and $\mathbb{C}^3 = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ is a product space, $g\colon Y \to X$ is continuous if and only if each of the composites

$$g_i\colon Y \xrightarrow{g} X \subset \mathbb{C}^3 \xrightarrow{\pi_i} \mathbb{C}$$

are continuous, for $i \in \{1, 2, 3\}$. Here

$$g_1(x, y, z) = x \ , \quad g_2(x, y, z) = x + y \quad \text{and} \quad g_3(x, y, z) = x + yz$$

are restrictions of polynomials, which are known to be continuous on $\mathbb{C}^3$, to the subspace Y. Hence each g_i is continuous.

1(b): The condition $|b - a| \neq |c - b| \neq |a - c| \neq |b - a|$ for a point $(a, b, c) \in X$ to lie in A is equivalent to the condition

$$|x + y - x| \neq |x + yz - (x + y)| \neq |x - (x + yz)| \neq |x + y - x|$$

for the corresponding point $(x, y, z) = f(a, b, c)$ in Y to lie in $f(A) = B$, since $(a, b, c) = g(x, y, z) = (x, x + y, x + yz)$. This simplifies to

$$|y| \neq |y||z - 1| \neq |y||z| \neq |y|\,,$$

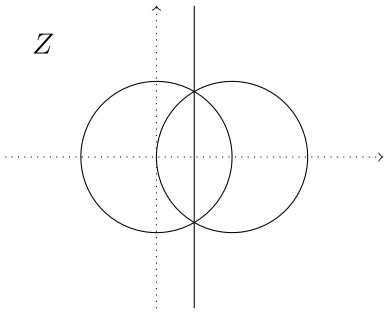
which is equivalent to the combined conditions

$$|y| \neq 0 \quad \text{and} \quad 1 \neq |z - 1| \neq |z| \neq 1\,.$$

Hence $(x, y, z) \in B$ if and only if $y \neq 0$ and

$$z \in Z = \{z \in \mathbb{C} \mid 1 \neq |z - 1| \neq |z| \neq 1\}\,.$$

We note that $0, 1 \notin Z$, so $Z \subset \mathbb{C} - \{0, 1\}$. Thus $B = \mathbb{C} \times (\mathbb{C} - \{0\}) \times Z$, with Z equal to the complement of the two circles and one vertical line shown below.



1(c): A homeomorphism induces a bijection between the sets of components, so A and B have the same number of components. The projection $\pi_3 \colon B \to Z$ maps each component of B to a component of Z, and for each component $K \subset Z$ the preimage $\pi_3^{-1}(K) = \mathbb{C} \times (\mathbb{C} - \{0\}) \times K$ is connected, since $\mathbb{C}$ and $\mathbb{C} - \{0\}$ are connected, and a product of connected spaces is connected. Hence B and Z have the same number of components. Inspection of the figure shows that Z has six components.

2(a): Let $j \colon Z \to Z$ denote the identity function. By the universal property of the subspace topology, $jp \colon X \to (Z, \mathcal{T}_2)$ is continuous because $f = i \circ jp \colon X \to Y$ is continuous. By the universal property of the quotient topology, $j \colon (Z, \mathcal{T}_1) \to (Z, \mathcal{T}_2)$ is continuous because $j \circ p \colon X \to (Z, \mathcal{T}_2)$ is continuous. Hence $U \in \mathcal{T}_2$ implies $U = j^{-1}(U) \in \mathcal{T}_1$, so $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

2(b): Let $X = \{a, b\}$ with the discrete topology and $Y = \{a, b\}$ with the trivial topology. Let $f \colon X \to Y$ be the identity function. Then f is continuous, and $Z = f(X) = \{a, b\}$. The quotient topology $\mathcal{T}_1$ is the discrete topology, and the subspace topology $\mathcal{T}_2$ is the trivial topology. In particular, $\mathcal{T}_1$ is strictly finer than $\mathcal{T}_2$.

2(c): Any continuous image of a compact space is compact, so X compact implies that $(Z, \mathcal{T}_1)$ is compact. Any subspace of a Hausdorff space is Hausdorff, so Y Hausdorff implies that $(Z, \mathcal{T}_2)$ is Hausdorff. Hence $j \colon (Z, \mathcal{T}_1) \to (Z, \mathcal{T}_2)$ is a continuous bijection from a compact space to a Hausdorff space, and is therefore a homeomorphism. Thus $\mathcal{T}_1 = \mathcal{T}_2$.

3: If $x, y \in X$ then $0 \le D(x, y) \le D(x, y) + D(y, x) \le D(x, x) = 0$ by (1) and (3). It follows that $D(x, y) = 0$, and $x = y$ by (1). Hence X cannot have two or more elements.

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