Model answers for the final exam of MAT3500/4500, fall 2022

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Convention:

- $\mathbb{R}$: the set of real numbers
- I: closed unit interval $[0, 1]$
- $\mathbb{Q}$: the set of rational numbers
- $\mathbb{C}$: the set of complex numbers
- S^1: unit circle in the Euclidean plane; $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$
- B^2: the closed unit disc in the Eucledean plane;
  $B^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$

We consider them as topological spaces in the conventional topology unless stated otherwise. (On $\mathbb{Q}$, as you can see below, we consider the induced topology for the inclusion $\mathbb{Q} \subset \mathbb{R}$.)

Part I

Problem 1. Consider the collection of subsets of $X = [-1, 1]$:

$$\mathcal{T} = \{X \cap [a, b) \mid a, b \in \mathbb{R}\}.$$

This cannot be interpreted as the collection of open sets for some topology on X. Choose an appropriate reason.

1. $(0, 1)$ is the union of $[1/n, 1)$, so $\mathcal{T}$ is not closed under union.
2. $[0, 1]$ is the complement of $[-1, 0)$ in X, so T is not closed under complement.
3. None of the above

Answer. 1.

Problem 2. Which of the following is closed in $\mathbb{R}$?

1. $A = \bigcap_{n=1}^{\infty} A_n$, where

   $$A_n = [0, 1/3^n] \cup [2/3^n, 3/3^n] \cup [2/3^{n-1}, 7/3^n] \cup \cdots \cup [(3^n - 1)/3^n, 1];$$

   A_n is the union of 2^n intervals of length $1/3^n$, and A is the intersection of the A_n.

2. $B = \mathbb{Q}$;

3. $C = \bigcup_{n=1}^{\infty} [0, 1 - 1/n]$; union of closed intervals of length $1 - 1/n$.

Answer. 1.

Problem 3. Which of the following is a base of the usual topology on $\mathbb{R}$?

1. $\mathcal{T} = \{[a, b) \mid a, b \in \mathbb{R}\}$
2. $\mathcal{T}' = \{A \subset \mathbb{R}\}$ (collection of all subsets)
3. None of the above

Answer. 3.

Problem 4. Let (X, d) be a metric space. Suppose that points x, y, z in X satisfy $d(x, y) = 1$ and $d(y, z) = 2$. What can we say about $d(x, z)$?

1. $1 \leq d(x, z) \leq 3$
2. $d(x, z) \geq 3$
3. $d(x, z) = 1$

Answer. 1.

Problem 5. Let X and Y be topological spaces. What can we say about the product topology on $X \times Y$?

1. Open sets are unions of the sets of the form $U \times V$, where U is open in X and V is open in Y.
2. Open sets are of the form $U \times V$, where U is open in X and V is open in Y.
3. None of the above

Answer. 1.

Problem 6. The real function

$$f(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0) \end{cases}$$

is not continuous. What is the appropriate reason for this?

1. Consider the open set $U = (1/2, 3/2)$ of $\mathbb{R}$. The inverse image $f^{-1}(U)$ is not open in $\mathbb{R}$.
2. Consider the open set $V = (-1/2, 1/2)$ of $\mathbb{R}$. The inverse image $f^{-1}(V)$ is not open in $\mathbb{R}$.
3. None of the above

Answer. 1.

Problem 7. Consider the following sequence of real functions f_n:

$$f_n(x) = \begin{cases} n|x| & (|x| \leq 1/n) \\ 1 & (|x| > 1/n) \end{cases}$$

What can we say about this?

1. This converges to $f(x) = 0$ or 1 according to $x = 0$ or not, in the topology of pointwise convergence.
2. This converges to $f(x) = 0$ or 1 according to $x = 0$ or not, in the topology of uniform convergence.
3. This converges to the constant function $f(x) = 1$ in any sensible topology.

Answer. 1.

Problem 8. Consider the following subsets of $\mathbb{R}^2$. Which one is connected for the induced topology?

1. $A = \{(x, y) \mid x^2 + y^2 \leq 1 \text{ or } (x-2)^2 + y^2 \leq 1\}$
2. $B = \{(x, 0) \mid x \in \mathbb{Q}\}$
3. $C = \{(x, \sin x) \mid x \in \mathbb{R}\}$

Answer. 1. or 3.

Problem 9. What can we say about the infinite product space $X = \prod_{i=1}^{\infty} \{0, 1\} = \{(x_i)_{i=1}^{\infty} \mid x_i = 0 \text{ or } 1\}$?

1. This space is compact, because it is the product of (copies of) the compact space $\{0, 1\}$.
2. This space is not compact, because it is not a closed bounded subset of $\mathbb{R}^k$.
3. None of the above

Answer. 1.

Problem 10. Which one is correct?

1. The loop in S^1 given by $f\colon I \to S^1, t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is not path-homotopic to the loop $g\colon I \to S^1, t \mapsto (\cos 4\pi t, \sin 4\pi t)$.
2. The loop in B^2 given by $f\colon I \to B^2, t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is not path-homotopic to the loop $g\colon I \to B^2, t \mapsto (\cos 4\pi t, \sin 4\pi t)$.
3. None of the above

Answer. 1.

Part II

Problem 11. Let $X = \prod_J \mathbb{R}$ be the direct product of uncountably many copies of $\mathbb{R}$, with the product topology. Thus, the index set J is an uncountable set. As a subspace of X, we consider

$$A = \{(x_i)_{i \in J} \mid x_i = 1 \text{ except for finitely many } i \in I\}.$$

1. Let x^0 denote the constant 0 sequence in X. (That is, $x^0 = (x_i^0)_i \in X$ defined by $x_i^0 = 0$ for all $i \in J$.) Show that x^0 belongs to the closure of A.
2. Let $(a^n)_{n=1}^\infty$ be a sequence in A, and for each n put
$$J_n = \{i \in J \mid a_i^n \neq 1\}.$$
Show that there is an element $\beta \in J$ not in $\bigcup_n J_n$.
3. Show that $(a^n)_n$ does not converge to x^0.

Answer. 1. We denote the projection map $X \to \mathbb{R}$ to the j-th factor by π_j. Let U be an open neighborhood of x^0. By definition of the product topology, there are finite number of indexes $j_1, \ldots, j_k$ and open sets $V_1, \ldots, V_k$ of $\mathbb{R}$ such that $B = \bigcap_{i=1}^k \pi_i^{-1}(V_i)$ satisfies $x^0 \in B \subset U$.

Define a point $y = (y_j)_j \in X$ by

$$y_j = \begin{cases} 0 & (j \in \{j_1, \ldots, j_k\}) \\ 1 & \text{otherwise.} \end{cases}$$

Then we have $y \in B$ from $y_{j_i} = 0 = x_{j_i} \in V_i$ for $i = 1, \ldots, k$. On the other hand, $y \in A$ because $y_j \neq 0$ happens only for $j = j_i$ for some i.

Thus, we obtain that an arbitrary neighborhood of x^0 contains a point of A.
2. Each J_n is a finite set by definition of A. Thus, their union $\bigcup_{n=1}^\infty J_n$ is (at most) a countable set. By assumption J is uncountable, hence there is $\beta \in J$ not in the union.
3. Convergence of $(a^n)_n$ to x^0 means that, given any (open) neighborhood U of X^0, there is an integer N such that $a^n \in U$ holds for any $n > N$.

Let $\beta \in J$ be as in the part 2. For any n, since $\beta \in J_n$, we have $a_\beta^n = 1$.

Define U as $\pi_\beta^{-1}((-1/2, 1/2))$. This is an open neighborhood of x^0. On the other hand, we have $a^n \notin U$ for all n by the above.

Problem 12. We say that a continuous map $f\colon X \to Y$ is *proper* if, for any compact subset $A \subset Y$, its inverse image $f^{-1}(A)$ is ~~proper~~ compact.

1. Give an example of a continuous map that is not proper.

Now, suppose that X and Y are locally compact spaces. Recall that we have one-point compactifications $X^+ = X \cup \{\infty\}$, $Y^+ = Y \cup \{\infty\}$ by formally adding extra points, and setting $(X \setminus K) \cup \{\infty\}$ for compact $K \subset X$ as open neighborhoods of ∞ in X^+, etc.

2. Given a continuous map $f\colon X \to Y$, consider the map $f^+\colon X^+ \to Y^+$ defined by
$$f^+(x) = \begin{cases} f(x) & (x \in X) \\ \infty & (x = \infty). \end{cases}$$
Show that f^+ is continuous when f is proper.

3. Find a (non-proper) continuous map f between locally compact spaces such that the above f^+ is not continuous.

Answer. 1. Take $X = \mathbb{R}$, $Y = \{*\}$, and $f\colon X \to Y$ be the only map. Then Y itself is compact, but its inverse image $f^{-1}(Y)$ is X itself. In the usual topology X is not compact, hence f is not proper.
2. We check that the inverse images $(f^+)^{-1}(U)$ are open in X^+ for open sets $U \subset Y^+$. There are two cases.

When U is an open subset of Y, its inverse image agrees with $f^{-1}(U)$ because $f^+(\infty) = \infty$ is not in U. By continuity of f, $f^{-1}(U)$ is open in X, hence it is also open in X^+.

Next suppose that $U = (Y \setminus K) \cup \{\infty\}$ for some compact $K \subset Y$. Then $(f^+)^{-1}(U)$ is equal to $(X \setminus f^{-1}(K)) \cup \{\infty\}$. Indeed, $f^+(\infty) = \infty$ implies $\infty \in (f^+)^{-1}(U)$. For the points in X, we have $f(x) \in U$ if and only if $f(x) \notin K$, that is, if and only if $x \in X \setminus f^{-1}(K)$.

By properness assumption $f^{-1}(K)$ is a compact subset of X. Thus, $(X \setminus f^{-1}(K)) \cup \{\infty\}$ is open in X^+.
3. Take $X = \mathbb{R}$, $Y = \{*\}$ as in part 1. Then X^+ is homeomorphic to the circle, while Y^+ consits of two points and has the discrete topology. The map f^+ maps the additional point ∞ to ∞, hence it is a surjective map from S^1 to $\{*, \infty\}$. Since continuous maps preserve connectedness, f^+ cannot be continuous.

Problem 13. We want to prove the following variation of the fundamental theorem of algebra: let $p(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients of degree $n > 0$, such that $|a_{n-1}| + \cdots + |a_0| < 1$. Then there is a complex number z such that $|z| < 1$ and $p(z) = 0$.

Identify S^1 with the subset $\{z \in \mathbb{C} \mid |z| = 1\}$ of $\mathbb{C}$, and similarly B^2 with $\{z \in \mathbb{C} \mid |z| \leq 1\}$. We also put $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

1. Assume that there is no z such that $|z| < 1$ and $p(z) = 0$. Show that the map $h\colon S^1 \to \mathbb{C}^\times, z \mapsto p(z)$ is homotopic to a constant map.
2. Under the same assumption, show that h is homotopic to the map $k\colon S^1 \to \mathbb{C}^\times, z \mapsto z^n$.
3. Derive a contradiction from the above.

Answer. 1. Consider the map $F\colon S^1 \times I \to \mathbb{C}^\times$ defined by $F(z, t) = p(tz)$. This is well-defined by assumption on p. At $t = 1$ we recover the original map given by p. At $t = 0$ we get the constant map $z \mapsto p(0)$.
2. This time consider the map $F'(z, t) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$. This is again well defined, because we have

$$\left|t(a_{n-1}z^{n-1} + \cdots + a_0)\right| \leq |a_{n-1}| + \cdots + |a_0| < 1 = |z^n|$$

by assumption. At $t = 1$ we recover the original map given by p. At $t = 0$ we get the map $k\colon z \mapsto z^n$.
3. Combining the above two, we get that the map k is homotopic to a constant map. However, the class of k in the fundamental group $\pi_1(S^1, *)$ corresponds to the integer n under the isomorphism $\pi_1(S^1, *) \simeq \mathbb{Z}$. This is a contradiction.

Problem 14. Given a commutative ring A, recall that we obtained a topological space

$$\operatorname{Spec} A = \{P \subset A \mid P \text{ is a prime ideal of } A\}$$

whose closed sets are given by

$$V(I) = \{P \in \operatorname{Spec} A \mid P \supset I\} \subset \operatorname{Spec} A.$$

1. Let $f\colon A \to B$ be a homomorphism of commutative rings. Show that the map $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A, P \mapsto f^{-1}(P)$ is continuous.

2. Given $P \in \operatorname{Spec} A$, let K be the field of fractions of the integral domain $O = A/P$. What is the continuous map $\operatorname{Spec} K \to \operatorname{Spec} A$ corresponding to the natural homomorphism $A \to K$?

3. Let A be a Noetherian ring, that is, a commutative ring where any ideal $I \subset A$ is finitely generated. Show that $\operatorname{Spec} A$ is compact.

Answer. 1. We prove that $(f^*)^{-1}(V(I))$ is closed in $\operatorname{Spec} A$ for all $I \subset A$. We show that $(f^*)^{-1}(V(I)) = V(f(I))$ holds.

Suppose $P \in V(f(I))$. Then $f^*(P) = f^{-1}(P)$ contains $f^{-1}(f(I))$, because taking inverse image preserves containment relation. Since $I \subset f^{-1}(f(I))$, we get $f^*(P) \in V(I)$.

On the other hand, suppose $Q \in \operatorname{Spec} B$ belongs to $(f^*)^{-1}(V(I))$. This means $f^*(Q) \in V(I)$, that is, $f^{-1}(Q) \supset I$. This implies $f(I) \subset Q$, that is, $Q \in V(f(I))$.

2. Since K is a field, it has only one prime ideal, namely $\{0\}$. The map $\operatorname{Spec} K \to \operatorname{Spec} A$ sends the unique element $x_0 \in \operatorname{Spec} K$ to the point $P \in \operatorname{Spec} A$. Indeed, the inverse image of $\{0\} \subset K$ in A agrees with P by construction.

3. Suppose that we have an open covering of $\operatorname{Spec} A$, that is, a collection of open sets $U_j = \operatorname{Spec} A \setminus V(I_j)$ for some indexes $j \in J$ such that $\bigcup_{j\in J} U_j = \operatorname{Spec} A$. We need to find a finite subcollection $J' \subset J$ such that $\bigcup_{j\in J'} U_j = \operatorname{Spec} A$.

We do not lose generality by assuming that each I_j is an ideal of A, hence $\bigcap_{j\in J} V(I_j)$ is equal to $V(\sum_{j\in J} I_j)$. The assumption on the I_j means $\bigcap_{j\in J} V(I_j) = \emptyset$, hence we get $V(I) = \emptyset$ for $I = \sum_{j\in J} I_j$. If I is a proper ideal of A, a maximal ideal containing I is an element of $V(I)$. This means $I = A$, hence 1 can be written as a (finite) sum $1 = a_1 + \cdots + a_k$ with $a_i \in I_{j_i}$ for some $j_i \in J$.

Take $J' = \{j_1, \ldots, j_k\}$, then we have $V(\sum_{j\in J'} I_j) = \emptyset$, hence we have found a desired J'.