MAT3500 / MAT4500 (2022 FALL) MANDATORY ASSIGNMENT

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Choose (at least) one of the problems from the list below, and submit your solutions via Canvas. The deadline is October 31.

Problem 1. Let $X = \mathbb{R}^2$, with the standard topology. We consider an equivalence relation $\sim$ on X defined by

$$(x, y) \sim (x', y') \Leftrightarrow y = y'.$$

What is the quotient space for this equivalence relation?

Problem 2. Let X be the disjoint union of two copies of closed disk. Formally, we can present it as

$$X = \{(i, x, y) \in \{0, 1\} \times \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 \leq 1\}.$$

Consider the equivalence relation on X given by $(0, x, y) \sim (1, x, y)$ when $x^2 + y^2 = 1$. (That is, $(i, x, y) \sim (j, v, w)$ only if $(x, y) = (v, w)$, and unless $x^2 + y^2 = 1$ we also impose $i = j$.) What is the quotient space for this equivalence relation?

Problem 3. Suppose we have a connected *simple graph* Γ (undirected graph without loops and parallel edges). Recall that a *path* in Γ is represented by a sequence of edges $e_1, e_2, \ldots, e_k$ such that e_i and e_{i+1} share a vertex for $1 \leq i < k$, and this k is called the *length* of the path. Let V be the set of vertices of Γ, and consider the function

$$d(v, v') = \min\{k \mid \text{there is a path of length } k \text{ between } v \text{ and } v'\} \quad (v, v' \in V).$$

Show that d is a metric on the set V.

Problem 4. Let X be the infinite product space $\prod_{i=1}^{\infty}\{0, 1\}$. Consider the metric on X given by

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} |x_i - y_i| \quad (x = (x_i)_{i=1}^{\infty}, y = (y_i)_{i=1}^{\infty}).$$

(1) Give a concrete choice of ϵ satisfying the following condition: $d(x, y) < \epsilon$ implies $x_i = y_i$ for $i = 1, 2$.
(2) Let N be an integer. Give a concrete choice of ϵ satisfying the following condition: $d(x, y) < \epsilon$ implies $x_i = y_i$ for $i = 1, \ldots, N$.
(3) Let $x^{(1)}, x^{(2)}, \ldots$ be a sequence of elements in X. (Overall we have a double sequence $x_i^{(j)}$ of 0's and 1's indexed by $i, j = 1, 2, \ldots$.) Suppose that $(x^{(j)})_j$ converges to y. What can be said about the sequence $(x_i^{(j)})_{i=1}^{\infty}$?

Problem 5. Let A be a commutative ring, and consider the set

$$\operatorname{Spec} A = \{P \subset A \mid P \text{ is a prime ideal of } A\}.$$

For each ideal $I \subset A$, consider the subset $V(I) \subset \operatorname{Spec} A$ defined by

$$V(I) = \{P \in \operatorname{Spec} A \mid P \supset I\}.$$

(1) Show that the collection

$$\mathcal{T}' = \{V(I) \mid I \text{ is an ideal of } A\}$$

satisfy the conditions for collection of closed sets. (That is, the collection

$$\mathcal{T} = \{\operatorname{Spec} A \setminus V(I) \mid I \text{ is an ideal of } A\}$$

defines a topology on $\operatorname{Spec} A$, called the *Zariski topology*.)
(2) Determine the open sets of $\operatorname{Spec} \mathbb{Z}$.
(3) Let x_0 be the point of $\operatorname{Spec} \mathbb{Z}$ represented by the ideal $\{0\} \subset \mathbb{Z}$. Is $\{x_0\}$ closed in $\operatorname{Spec} \mathbb{Z}$?

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