Let $l \subset S$ be a line in an incidence geometry satisfying betweenness axioms. Define a relation $\sim$ on $S-l$ by

$A \sim B \Leftrightarrow AB \cap l = \phi$ (A, B are the same side of l)

<u>Claim</u>: $\sim$ is an equivalence relation on $S-l$, with exactly two equivalence classes. ("<u>Line separation property</u>"

<u>Proof</u> - $A \sim A$ trivial (note that $AA = \{A\}$).
- $A \sim B \Rightarrow B \sim A$, since $AB = BA$

Need to prove <u>transitivity</u>:
<u>$A \sim B$ & $B \sim C \Rightarrow A \sim C$</u>.

Only nontrivial case is if A. B. C are <u>distinct</u>

<u>Case 1</u>: A, B, C not on the same line. Then it follows from B4, since if $AC \cap l \neq \phi$, then $AB \cap l \neq \phi$ or $BC \cap l \neq$,

<u>Case 2</u>: ABC are the same line m
Then clearly $m \neq l$, since $A \in m$, $A \notin l$

If $m \cap l = \phi$ we are done, since $AC \subset m$
Otherwise, let $m \cap l = \{P\}$ (one point, by I
By I2, we choose another point $Y \in l$ ($Y \neq P$,

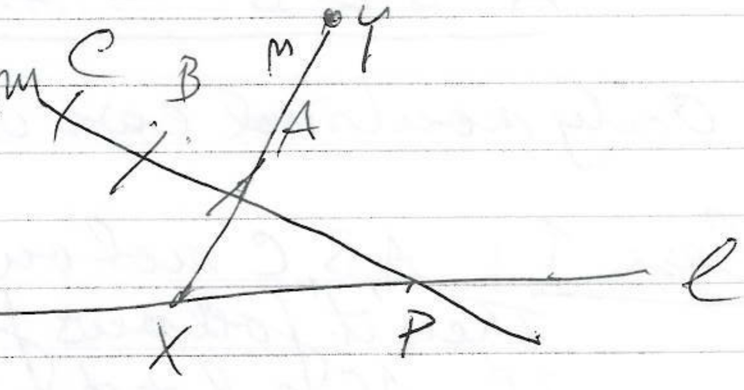
Let n be the line $\overline{XA}$, and choose $Y \in n$ such that $X*A*Y$. Then $AY \cap l = \emptyset$, since if $A*W*Y$, $W \in l$, then $W \in l \cap n = \{X\}$ but we can not have both $A*X*Y$ and $X*A*Y$ by B3. It follows that $A \sim Y$.

Now $n \neq m$ ($X \in n$ but not on m), so A, Y, B do not lie on a line. Therefore also $B \sim Y$, by case 1 (since $A \sim B$ and $A \sim Y$).

Similarly $C \sim Y$, since B, C and Y do not lie on a line and $B \sim C$ & $B \sim Y$.

Since A, C and Y also do not lie on a line, it now follows that also $A \sim C$.

Illustration:



Two equivalence classes:

At least two: I3 $\Rightarrow \exists A \notin l$.
Let $P \in l$, by B2 $\exists B$ s.t. $A*P*B$. Hence $A \not\sim B$, so A and B are in distinct equivalence classes.

Alternative two: Given $A, B, B' \notin l$ s.t. $A \not\sim B$, $A \not\sim B'$, then we must prove that $B \sim B'$.

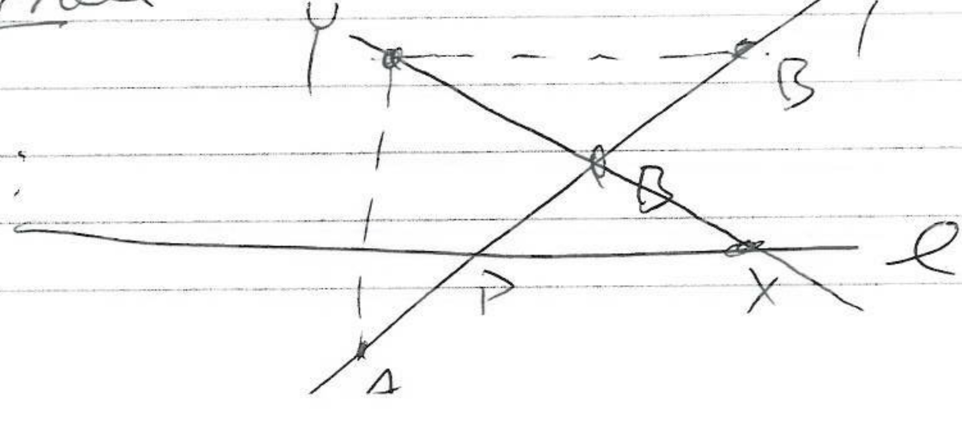
Case 1 A, B, B' not on a line.
Since $AB \cap l \neq \phi$ and $AB' \neq \phi$, it follows from the uniqueness of B4 that $BB' \cap l = \phi$ — i.e. $B \sim B'$.

Case 2 A, B, B' on a line m (and $B \neq B'$)
By assumption, m & l intersect in a point P (unique $\Rightarrow$ $A * P * B$ and $A * P * B'$)
Let $X \in l$, $X \neq P$ and choose Y s.t. $X * B * Y$
Then, as in Case 2 above, we have $B \sim Y$.
Since $A \not\sim B$, it follows that $A \not\sim Y$ — i.e. $AY \cap l \neq \phi$. In the triangle A, Y, B' l then intersects two sides AY and AB', hence $YB' \cap l = \phi$ by B4.

Since now $B \sim Y$ and $Y \sim B'$, we have $B \sim B'$

$\boxed{}$

Illustration:



Consequence: If $A*B*C$ and $B*C*D$ then $A*B*D$ and $A*C*D$.

Proof. All the points must lie on a line m. Let l be another line, such that $l \neq m$ and $l \cap m = \{C\}$. (Follows by I1 & I3).
Let $\sim$ mean "same side of l". Then $B*C*D$ implies $B \not\sim D$

But $AB \cap l = \phi$, since m & l only cross at C and we cannot have $A*C*B$ by B3. Hence $A \sim B \Rightarrow A \not\sim D$, ie there is a P s.t. $A*P*D$, $P \in l$.

But then $P \in l \cap m \Rightarrow \underline{\underline{P = C}}$

($A*B*D$ completely similar).