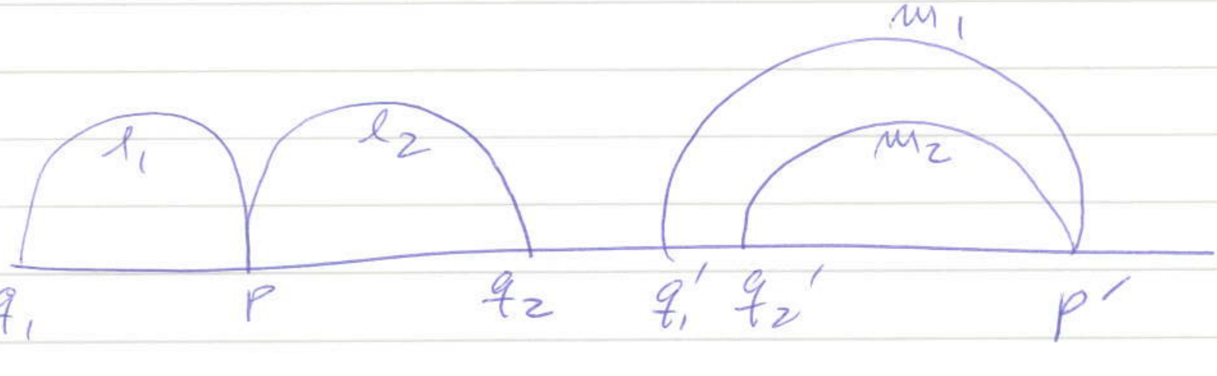
2.2.7 a)

Given two triples (z_1, z_2, z_3) and (w_1, w_2, w_3) in $\bar{\mathbb{R}}$. By Cor. 2.2.5 there exists a fractional linear transformation $f(z)$ such that $f(z_i) = w_i$, and f has real coefficients. However, f may not lie in Möb(H), since it may have determinant $(ad - bc)$ negative and map the upper halfplane to the lower halfplane (equation 2.2.9). By composing with complex conjugation, we get an element in Möb(H).

If (l_1, l_2) and (m_1, m_2) are two pairs of H-lines, apply this to the two triples of <u>endpoints</u> of the lines:



If $f \in$ Möb(H) maps p to p' and q_i to q_i', it has to map l_i to m_i $(i = 1, 2)$.

b) If (z_1, z_2), (w_1, w_2) are two such pairs, the obvious thing to try is to use a) by choosing auxiliary points z_3 and w_3

and apply a). However, this may take us outside $Möb^+(\mathbb{H})$, if z_3, w_3 are chosen arbitrarily.

From the proof of Cor. 2.2.5 we see that it suffices to choose z_3, w_3 such that the FLT's h and g have positive determinant. But by the formulas in Lemma 2.2.3, this can be achieved by choosing z_3 and w_3 appropriately.

c) The simplest is to find an example of two pairs of points that can not be mapped to each other by an element in $Möb(\mathbb{H})$.

Let the pairs of points be $(i, 2i)$ and $(i, 3i)$ and suppose $f(i) = i$, $f(2i) = 3i$.

If $f \in Möb^+(\mathbb{H})$, $f(z) = \frac{az+b}{cz+d}$, and then we get

$$ai+b = i(ci+d) \Rightarrow a=d,\ b=-c$$

$$a \cdot 2i + b = 3i(c\,2i+d) \Rightarrow 2a=3d,\ b=-6c$$

But then $a=b=c=d=0$, which is impossible.

If $f \in Möb^-(\mathbb{H})$, $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, which gives similar equations with only the zero solution.

Hence no such f can exist.

2.2.9 We can write $g(z) = f(\bar{z})$, where f is a fractional linear transformation.

Then we have, by 2.2.10 (iii),

$$[g(z), g(z_1) g(z_2) g(z_3)] = [\bar{z}, \bar{z}_1, \bar{z}_2, \bar{z}_3]$$

But by the formulas in the proof of lemma 2.2.3 the latter is equal to

$$\overline{[z, z_1, z_2, z_3]}.$$

2.2.10 Let $GL_2^+(\mathbb{R})$ and $GL_2^-(\mathbb{R})$ be the (invertible) 2×2-matrices with positive and negative determinant. Then

$$GL_2(\mathbb{R}) = GL_2^+(\mathbb{R}) \cup GL_2^-(\mathbb{R}),$$

with $GL_2^+(\mathbb{R})$ a subgroup of index 2 and $GL_2^-(\mathbb{R})$ the nontrivial coset.

Note that the matrix $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in GL_2^- \mathbb{R}$, and $GL_2^-(\mathbb{R}) = T\, GL_2^+(\mathbb{R})$.

This is completely analogous to the decomposition

$$\text{Möb}(H) = \text{Möb}^+(H) \cup \text{Möb}^-(H),$$

where $\text{Möb}^+(H)$ is a subgroup of index 2 and $\text{Möb}^-(H)$ is the coset containing the map $\eta(z) = -\bar{z}$.

Now define $\phi : GL_2(\mathbb{R}) \to \text{Möb}(\mathbb{H})$
by
$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases} z \mapsto \dfrac{az+b}{cz+d} & \text{if } ad-bc>0 \\ z \mapsto \dfrac{a\bar{z}+b}{c\bar{z}+d} & \text{if } ad-bc<0 \end{cases}$$

Then ϕ is surjective by Prop. 2.2.9, and a calculation shows that it is a homomorphism.

(We know that $\phi \mid GL_2^+(\mathbb{R})$ is a homomorphism, and thus it suffices to show that
$\phi(T) = \eta$ and that
$\phi(TA) = \eta\, \phi(A)$, $\phi(AT) = \phi(A) \cdot \eta$)

It now only remains to observe that
$$\ker \phi = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \,\middle|\, \lambda \neq 0 \right\}$$