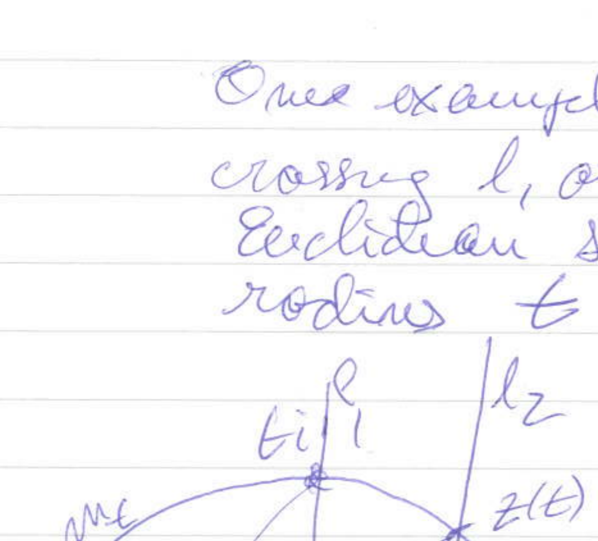
2.5.2 Clearly $d(U, V) = d(gU, gV)$ if g is an isometry. Therefore we can assume that l_1 is the imaginary axis and l_2 a vertical line $\{c + ti;\ t > 0\}$ with $c > 0$. It then suffices to find points $z(t) \in l_2$ such that

$$d(ti, z(t)) \to 0 \text{ as } t \to \infty .$$

One example: Let m_t be the H-line crossing l_1 orthogonally at ti – i.e. a Euclidean semicircle with center at 0 and radius t. Let $z(t)$ be the intersection between m_t and l_2. (Exists if $t > c$.)



Then $z(t) = c + \sqrt{t^2 - c^2}\, i$, and (cf. fig 2.5.1)

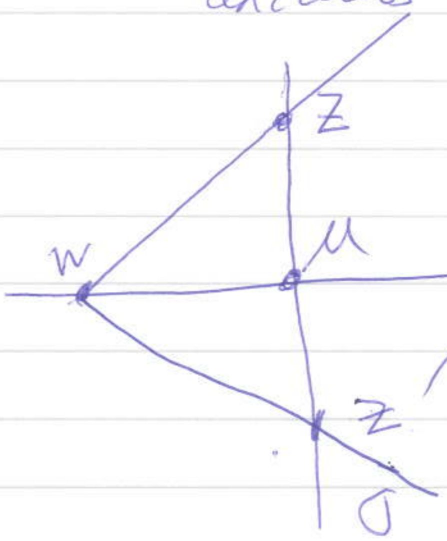
$$d(ti, z(t)) = \left| \ln \left| \frac{\tan A}{\tan B} \right| \right|$$

$$= \left| \ln \left(\frac{\sqrt{t^2 - c^2}}{t + c} \right) \right| = \left| \ln \sqrt{\frac{t-c}{t+c}} \right| \to 0$$

as $t \to \infty$.

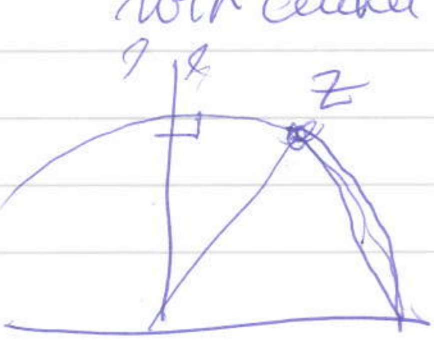
(Remark. Using instead the formula (2.7.6), this is even simpler, using $z(t) = c + it$.)

2.6.1 This can be solved in many ways. The most geometric proof uses only axioms I1–3, B1–4 and C1–6:

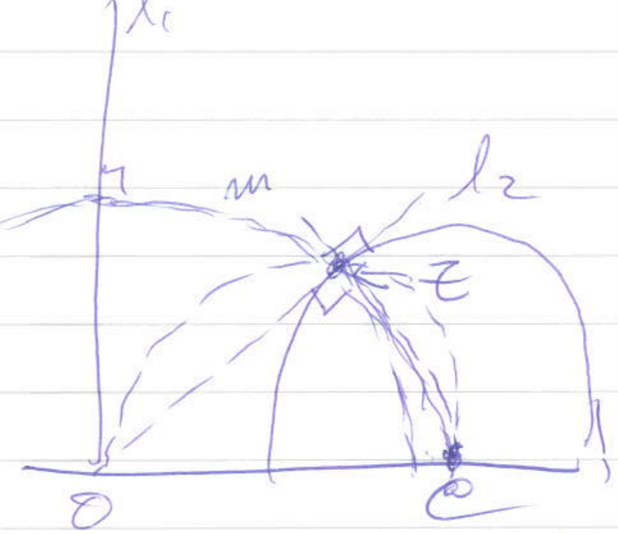


- choose $w \in l$ and draw ray $\overrightarrow{w,z}$. If this is not orthogonal to l, choose a second point on l and construct ray σ on opposite side of l such that $\angle w'w\sigma \cong \angle w'wz$.
- ~~Find~~ point z' on σ s.t. $[w,z'] \cong [w,z]$, and draw the line $\overleftrightarrow{zz'}$. This must cross l in u, say.
- Now, by C6, the triangles $\triangle zwu$ and $\triangle z'wu$ are congruent — in particular $\angle wuz \cong \angle wuz'$. But these angles are also supplementary, hence right, by definition.

Another proof goes by moving l to the imaginary axis. Then a line through z orthogonal to l is a Euclidean semicircle with center at 0, as in figure. But this has radius $|z|$ (Euclidean norm)

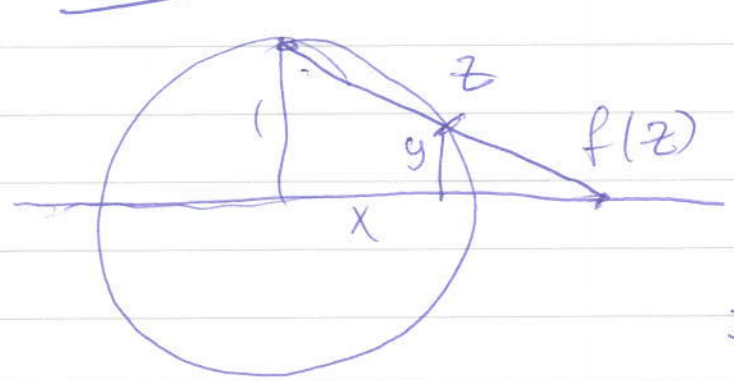


2.6.2 Move l_1 to the imaginary axis (cf. fig).



By 2.6.1 it suffices to find the point z of intersection between l_2 and the line m we want to determine. But since l_2 and m meet orthogonally, the tangent of l_2 at z is a radius of l_2, which then must meet a radius of l_2 at z. But then z also lies on a circle with diameter $[OC]$, where c is the center of l_2.

2.7.1 Stereographic projection from circle:



Similar triangles give $\dfrac{f(z)-x}{f(z)} = \dfrac{1}{y}$

$$\Rightarrow f(z) = \frac{x}{1-y}$$

$$G^{-1}(z) = \frac{i(x+iy)-1}{-x-iy+i} = \frac{-(y+1)+ix}{-x-(y-1)i} = \frac{y+1-ix}{x+(y-1)i}$$

$$= \dots = \frac{2x-(x^2+y^2-1)i}{x^2+y^2-2y+1} \underset{\uparrow}{=} \frac{2x}{2-2y} = f(z)$$

Standard manipulations

when $x^2+y^2=|z|^2=1$

<u>2.7.3</u> Fixpoints for $g(z) = \frac{\alpha \bar{z} + \beta}{\bar{\beta}\bar{z} + \bar{\alpha}}$:

$$\alpha \bar{z} + \beta = \bar{\beta}|z|^2 + \bar{\alpha} z$$

$$\bar{\beta}|z|^2 - \underbrace{(\alpha \bar{z} - \bar{\alpha} z)}_{\substack{\shortparallel \\ 2\,\mathrm{Im}\,\alpha\bar{z}\cdot i}} - \beta = 0$$

Separating real and imaginary parts gives two equations:

$$\begin{cases} \mathrm{Re}\,\beta\,(|z|^2 - 1) = 0 \\ \mathrm{Im}\,\beta\,|z|^2 + 2\,\mathrm{Im}\,\alpha\bar{z} + \mathrm{Im}\,\beta = 0 \end{cases}$$

<u>Case 1</u> $\mathrm{Re}\,\beta \neq 0$. Then $|z|^2 = 1$, so there are no fixpoints in $\mathbb{D}$.

<u>Case 2</u> $\mathrm{Re}\,\beta = 0$ Then there is only the second equation, which gives a $\mathbb{D}$-line.

Hence this happens if and only if <u>$\mathrm{Re}\,\beta = 0$</u>

Then <u>$\bar{\beta} = -\beta$</u>, and we can write

$$\frac{\alpha\bar{z} + \beta}{\bar{\beta}\bar{z} + \bar{\alpha}} = \frac{\alpha}{\bar{\beta}} + \frac{\beta - \frac{\alpha\bar{\alpha}}{\bar{\beta}}}{\bar{\beta}\bar{z} + \bar{\alpha}} = \frac{\alpha}{\bar{\beta}} + \frac{|\beta|^2 - |\alpha|^2}{|\beta|^2\left(\bar{z} + \frac{\bar{\alpha}}{\bar{\beta}}\right)}$$

$$= \frac{\alpha}{-\beta} + \frac{\frac{|\alpha|^2 - |\beta|^2}{|\beta|^2}}{\left|z - \left(\frac{\alpha}{-\beta}\right)\right|^2}\left(z - \frac{\alpha}{-\beta}\right)$$

This is the general form for an inversion (2.3.3) with $m = \frac{\alpha}{-\beta}$, $r^2 = \frac{|\alpha|^2 - |\beta|^2}{|\beta|^2}$.