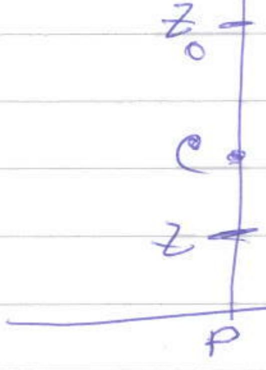
2.7.5 Since isometries trivially map hyperbolic circles to hyperbolic circles, it suffices to consider the case where l is a vertical line in $\mathbb{H}$ and the end-point P on the real axis. If the center



c lies on the ray $[z_0, P)$, there is also another point z on the circle such that $z_0 * c * z$ and $d_H(z_0, c) = d_H(z, c)$ (axiom C.1). As $c \to P$, we must also have $z \to P$.

But hyperbolic circles are also Euclidean circles (Exc. 2.7.4 – proved in lecture – follows most easily by moving to $\mathbb{D}$ and center at $0 \in \mathbb{D}$). So the limiting position must be a circle in $\mathbb{C}$ which goes through z_0 and P and lies in $\mathbb{H} \cup \mathbb{R}$. The only possibility is a circle tangent to $\mathbb{R}$, i.e. a horocircle.

2.7.6 To distinguish between the hyperbolic an Euclidean situations, let us use superscripts h and e. E.g. $(\mathbb{D}, d^h)$, $(\mathbb{D}^e, d^e)$ for the metric spaces and $\mathcal{T}^h, \mathcal{T}^e$ for the topologies.

The explicit formula for d^h (2.7.5) shows immediately that $d^h : \mathbb{D}^e \times \mathbb{D}^e \to [0, \infty)$

is continuous. The natural identification

$$B^h_\varepsilon(x) = (d^h)^{-1}([0,\varepsilon)) \cap \{x\}\times \mathbb{D}$$

then shows that the ε-neighborhoods in $\mathcal{T}^h$ also are open in $\mathcal{T}^e$ – hence $\mathcal{T}^e$ is finer than $\mathcal{T}^h$.

Now use the formula $d^h(z_1,z_2) = \ln\left(\frac{1+\rho}{1-\rho}\right)$, where $\rho = \left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| \in [0,1)$

To estimate this, observe first that

$$\frac{d}{dx}\left(\ln\left(\frac{1+x}{1-x}\right)\right) = \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2} \geq 2$$

for $|x|<1$

It follows that

$$\ln\left(\frac{1+x}{1-x}\right) = \left|\ln\left(\frac{1+x}{1-x}\right) - \ln\left(\frac{1+0}{1-0}\right)\right| \geq 2\cdot x \text{ for } 0\leq x<1$$

But since $|z_1|, |z_2|$ both are <1, $|1-\bar{z}_1 z_2| \leq 1 + |\bar{z}_1 z_2| < 2$

$$\Rightarrow d^h(z_1,z_2) \geq 2\frac{|z_1-z_2|}{|1-\bar{z}_1 z_2|} \geq |z_1-z_2| \text{ for } z_1,z_2\in\mathbb{D}$$

Then we have $B^h_\varepsilon(z) \subset B^e_\varepsilon(z)\ \forall z,\varepsilon$, so $\mathcal{T}^h$ is also finer than $\mathcal{T}^e$

28.2 Write $g(z) = u(z) + i v(z)$; $z = (x + iy)$, thinking of g as a function $\mathbb{C} \to \mathbb{C}$, and

$$g(x,y) = (u(x,y), v(x,y))$$

If we think of g as a function $\mathbb{R}^2 \to \mathbb{R}^2$

Then $J(g) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$

But if g is complex analytic, the Cauchy-Riemann equations say that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, so

$$|J(g)| = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

But g complex analytic means that the derivatives are the same in all directions. In particular

$$g'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow |g'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |J(g)|.$$