5.1.3

$$\begin{array}{ccc} S & \xrightarrow{f} & S' \\ \uparrow x & & \uparrow y \\ U & & V \\ \text{coord.}(u_1, u_2) & & (v_1, v_2) \end{array}$$

functions of (u_1, u_2)

Let $g^{-1} f \cancel{x} = (g_1, g_2)$ — v_1, v_2

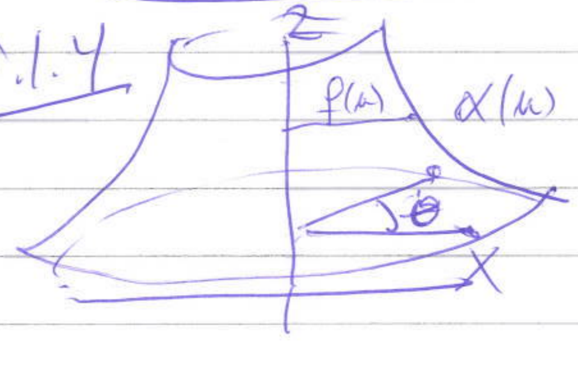
$$df(x_{u_1}) = dy\left(d(g^{-1} f x)(1,0)\right)$$

$$= dy\left(\frac{\partial g_1}{\partial u_1}, \frac{\partial g_2}{\partial u_1}\right) = \frac{\partial g_1}{\partial u_1} y_{v_1} + \frac{\partial g_2}{\partial u_1} y_{v_2}$$

$$df(x_{u_2}) = \cdots \qquad = \frac{\partial g_1}{\partial u_2} y_{v_1} + \frac{\partial g_2}{\partial u_2} y_{v_2}$$

But $J(g^{-1} f x) = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix}$.

5.1.4



$$x(u, \theta) = \left(f(u)\cos\theta, f(u)\sin\theta, g(u)\right)$$
$$\theta \in \mathbb{R}$$

$$x_u = \left(f'(u)\cos\theta, f'(u)\sin\theta, g'(u)\right)$$
$$x_\theta = \left(-f\sin\theta, f\cos\theta, 0\right)$$

– lin indep $\Leftrightarrow$ $f' f \neq$ or $g' f \neq 0$.

$\Leftrightarrow$ $\alpha'(u) \neq 0$

$f < 0$ – similar, $f = 0$ somewhere $\Rightarrow$ not loc Eucl.
– except $g = \text{const} \Rightarrow$ plane.

5.3.1 ((cf. 5.1.4).

$$E = X_u \cdot X_u = f'(u)^2 + g'(u)^2 = |\alpha'(u)|^2$$

$$F = X_u \cdot X_\theta = 0 \quad , \quad G = X_\theta \cdot X_\theta = f(u)^2$$

5.3.3 Manipulating formally with matrix notation, we can write

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle X_u, X_u\rangle, & \langle X_u, X_v\rangle \\ \langle X_u, X_u\rangle, & \langle X_v, X_v\rangle \end{bmatrix} = \left\langle \begin{bmatrix} X_u \\ X_v \end{bmatrix}, [X_u \ X_v] \right\rangle$$

$$= \left\langle \begin{bmatrix} X_u \\ X_v \end{bmatrix}, \begin{bmatrix} X_u \\ X_v \end{bmatrix}^t \right\rangle, \text{ etc.}$$

From Ex 5.1.3 with $f = id$ and x, y switched, we have

$$\begin{bmatrix} y_s \\ y_t \end{bmatrix} = \begin{bmatrix} \frac{\partial x_u}{\partial y_s} X_u + \frac{\partial x_v}{\partial y_s} X_v \\ \frac{\partial x_u}{\partial y_t} X_u + \frac{\partial x_v}{\partial y_t} X_v \end{bmatrix} = J(x^{-1}y)^t \begin{bmatrix} X_u \\ X_v \end{bmatrix}$$

Then $\begin{bmatrix} E' & F' \\ F' & G' \end{bmatrix} = \left\langle J(x^{-1}y)^t \begin{bmatrix} X_u \\ X_v \end{bmatrix}, \left(J(x^{-1}y)^t \begin{bmatrix} X_u \\ X_v \end{bmatrix}\right)^t \right\rangle$

$$= \left\langle J(x^{-1}y)^t \begin{bmatrix} X_u \\ X_v \end{bmatrix}, \begin{bmatrix} X_u \\ X_v \end{bmatrix}^t J(x^{-1}y) \right\rangle$$

$$= J(x^{-1}y)^t \begin{bmatrix} E & F \\ F & G \end{bmatrix} J(x^{-1}y)$$

3.1.3

$$\begin{tikzcd} S \arrow[r, "f"] & S' \\ U \arrow[u, "x"] & V \arrow[u, "y"'] \end{tikzcd}$$

coord. (u_1, u_2) $\qquad$ (v_1, v_2)

Let $g^{-1} f \cancel{x} = (g_1, g_2)$ $\qquad$ functions of (u_1, u_2) / $v_1\ v_2$

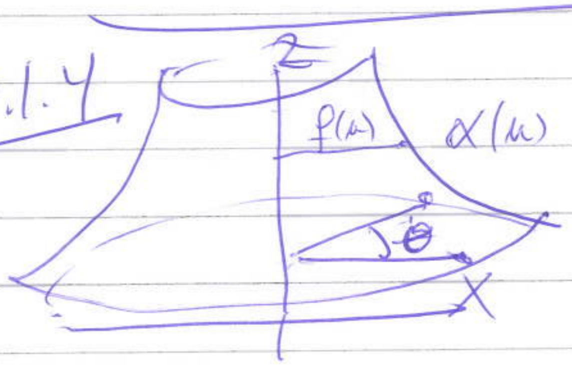
$$df(x_{u_1}) = dy\left(d(g^{-1} f x)(1,0)\right)$$

$$= dy\left(\frac{\partial g_1}{\partial u_1}, \frac{\partial g_2}{\partial u_1}\right) = \frac{\partial g_1}{\partial u_1} y_{v_1} + \frac{\partial g_2}{\partial u_1} y_{v_2}$$

$$df(x_{u_2}) = \cdots = \frac{\partial g_1}{\partial u_2} y_{v_1} + \frac{\partial g_2}{\partial u_2} y_{v_2}$$

But $J(g^{-1} f x) = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix}$.

3.1.4



$x(u, \theta) = (f(u)\cos\theta, f(u)\sin\theta, g(u))$

$\theta \in \mathbb{R}$

$x_u = (f'(u)\cos\theta, f'(u)\sin\theta, g'(u))$

$x_\theta = (-f\sin\theta, f\cos\theta, 0)$

– lin indep $\Leftarrow$ $f' f \neq$ or $g' f \neq 0$.

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5.3.1 ((cf. 5.1.4).

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5.3.3 Manipulating formally with matrix notation, we can write

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle x_u, x_u\rangle, & \langle x_u, x_v\rangle \\ \langle x_u, x_u\rangle, & \langle x_v, x_v\rangle \end{bmatrix} = \left\langle \begin{bmatrix} x_u \\ x_v \end{bmatrix}, [x_u \; x_v] \right\rangle$$

$$= \left\langle \begin{bmatrix} x_u \\ x_v \end{bmatrix}, \begin{bmatrix} x_u \\ x_v \end{bmatrix}^t \right\rangle, \text{ etc.}$$

From Ex 5.1.3 with $f = id$ and x, y switched, we have

$$\begin{bmatrix} y_s \\ y_t \end{bmatrix} = \begin{bmatrix} \frac{\partial x_u}{\partial y_s} x_u + \frac{\partial x_v}{\partial y_s} x_v \\ \frac{\partial x_u}{\partial y_t} x_u + \frac{\partial x_v}{\partial y_t} x_v \end{bmatrix} = J(x^{-1}y)^t \begin{bmatrix} x_u \\ x_v \end{bmatrix}$$

Then $$\begin{bmatrix} E' & F' \\ F' & G' \end{bmatrix} = \left\langle J(x^{-1}y)^t \begin{bmatrix} x_u \\ x_v \end{bmatrix}, \left(J(x^{-1}y)^t \begin{bmatrix} x_u \\ x_v \end{bmatrix}\right)^t \right\rangle$$

$$= \left\langle J(x^{-1}y)^t \begin{bmatrix} x_u \\ x_v \end{bmatrix}, \begin{bmatrix} x_u \\ x_v \end{bmatrix}^t J(x^{-1}y) \right\rangle$$

$$= J(x^{-1}y)^t \begin{bmatrix} E & F \\ F & G \end{bmatrix} J(x^{-1}y)$$

5.3.4 (i) From 5.3.3 it follows that if E, F, G are smooth and $x^{-1}y$ is smooth, then E', F', G' are smooth.

(ii) Take determinants in 5.3.3 and use that $\det(A) = \det(A^t)$.

(iii) It follows that if R is parametized by Ω in x-coord and Ω' in y-coord, then

$$\iint_{\Omega'} \sqrt{E'G'-F'^2}\, ds\, dt = \iint_{\Omega'} \sqrt{EG-F^2}\, |J(x^{-1}y)|\, ds\, dt$$

$$= \iint_{\Omega} \sqrt{EG-F^2}\, du\, dv .$$