5.4.5

a) $\mathcal{U} = \mathbb{R} \times (0, \infty)$.

$X : \mathcal{U} \to \mathbb{H}$

$X(u,v) = u + \cosh v\, i$ $\qquad$ $X(\mathcal{U}) = M$

- smooth and invertible, since $\cosh v$ is invertible for $v > 0$.

$X_u = (1, 0)$, $X_v = (0, \sinh v)$

$E = \frac{1}{(\cosh v)^2}$ $F = 0$, $G = \frac{(\sinh v)^2}{(\cosh v)^2} = (\tanh v)^2$

b) $y : \mathcal{U} \to \mathbb{R}^3$,

$y(u,v) = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, v - \tanh v\right)$

$y_u = \left(-\frac{\sin u}{\cosh v}, \frac{\cos u}{\cosh v}, 0\right)$

$y_v = \left(-\frac{\sinh v \cos u}{\cosh^2 v}, -\frac{\sinh v \sin u}{\cosh^2 v}, \underbrace{1 - \frac{1}{\cosh^2 v}}_{= \tanh^2 v}\right)$

Then $E = \frac{1}{\cosh^2 v}$, $F = 0$, $G = \frac{\sinh^2 v}{\cosh^4 v} + \frac{\sinh^4 v}{\cosh^4 v}$

$G = \frac{\sinh^2 v (1 + \sinh^2 v)}{\cosh^4 v} = \frac{\sinh^2 v \cdot \cosh^2 v}{\cosh^4 v} = \tanh^2 v$

Since $|y_u \times y_v| = \sqrt{EG}$ clearly is nonzero for all $(u,v) \in \mathcal{U}$, y defines a regular parametrization on a neighborhood of every point. Hence $\Sigma = y(\mathcal{U})$ is a regular surface.

c) Observe that $y(u,v) = y(u',v')$ if and only if $v = v'$, $u - u' = 2k\pi$ for some integer k.
If we define the Möbius transformation $\gamma \in \text{Möb}^+ H$ by $\gamma(z) = z + 2\pi$, we see that $X(u,v) = X(u',v') \iff y(u,v) = \gamma^k y(u',v')$ for some integer ~~k~~. Hence $X y^{-1}$ defines a diffeomorphism between the surfaces M/Γ and Σ. Moreover, since γ is an isometry of M to itself, M/Γ inherits a hyperbolic and Riemannian structure such that $y: \mathcal{U} \to M \to M/\Gamma$ defines local parametrizations. But since X and y give the same functions E, F, G, these surfaces are also isometric.

5.5.1

$$X_u = (g'\cos v, g'\sin v, h')$$
$$X_v = (-g\sin v, g\cos v, 0)$$
$$X_{uu} = (g''\cos v, g''\sin v, h'')$$
$$X_{uv} = (-g'\sin v, g'\cos v, 0)$$
$$X_{vv} = (-g\cos v, -g\sin v, 0)$$

$$\Rightarrow \quad X_u \times X_v = (-h'g\cos v, -h'g\sin v, g'g)$$
$$EG-F^2 = (g'^2+h'^2)g^2$$
$$N = \frac{(-h'\cos v, -h'\sin v, g')}{\sqrt{g'^2+h'^2}}$$

This gives $e = N \cdot X_{uu} = \dfrac{-h'g''\cos^2 v + (-h'g''\sin^2 v) + g'h''}{\sqrt{g'^2+h'^2}}$

$$= \frac{-h'g''+g'h''}{\sqrt{g'^2+h'^2}}$$

Similarly: $f = N \cdot X_{uv} = 0$
$$g = N \cdot X_{vv} = gh' / \sqrt{g'^2+h'^2}$$

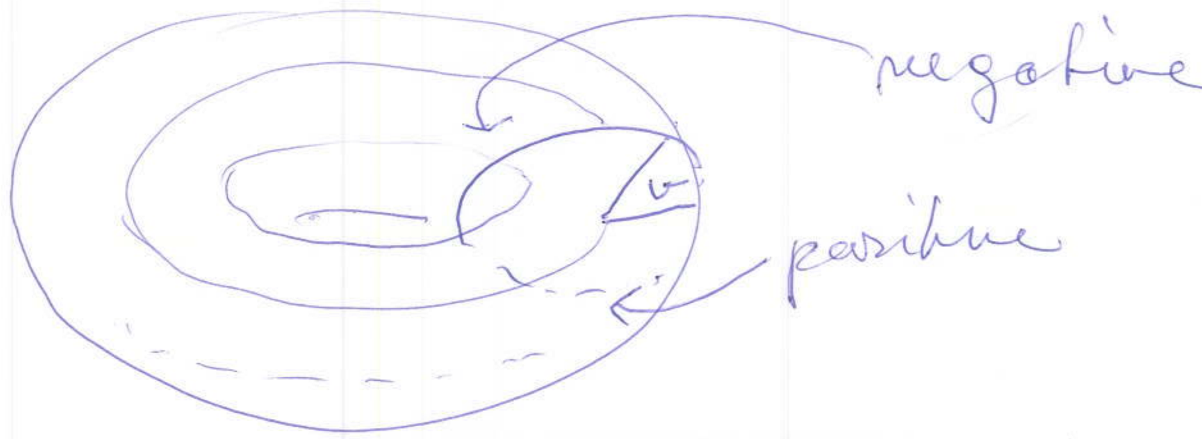
Hence, $K = \dfrac{eg-f^2}{EG-F^2} = \dfrac{(g'h''-h'g'') \cdot gh'}{(g'^2+h'^2)^2 g^2} = \dfrac{(g'h''-h'g'')h'}{(g'^2+h'^2)^2 g}$

5.5.2 Substitute $\begin{cases} g(v) = a\cos v + b \\ h(v) = a \sin v \end{cases}$ in 5.5.1

Then
$$K(r(u,v)) = \frac{a\cos v\left(-a\sin v(-a\sin v) - (-a\cos v)\, a\cos v\ldots\right.}{(a\cos v + b)(a^2\cos^2 v + a^2 \sin^2 v)}$$
$$= \frac{a^3 \cos v}{a\cos v + b}.$$

Under the conditions $0 < a < b$, the denominator is always positive, so the sign depends on $\cos v$.

Hence, $K > 0$ for $v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $K < 0$ for $v \in \left(\frac{\pi}{2}, \frac{3\ldots}{2}\right.$



5.5.3 If $g'^2+h'^2=1$, we get, by derivation

$$h'h''+g'g''=0, \text{ or } \underline{h'h''=-g'g''}$$

Then, from 5.5.1:

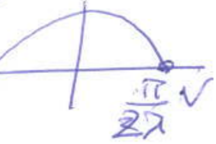
$$K=\frac{g'h'h''-g''h'^2}{g\cdot 1}=\frac{-g'^2g''-g''h'^2}{g}=-\frac{g''}{g}$$

If K is constant, the differential equation $K=-\frac{g''}{g}$ is easy to solve for g, and then $h(v)=\int_a^v\sqrt{1-g'(t)^2}\,dt$ give all solutions for h, as a varies.

If $K\le 0$, there are no compact such surfaces. So, let $K=\lambda^2>0$, $\lambda>0$.

Then $g(v)=A\cos(\lambda v+\beta)$, $A>0$, and we can choose $\beta=0$ (by translation?).

Note that then $g(v)=g(-v)$ and $g(v)>0$ for $\lambda v\in(-\frac{\pi}{2},\frac{\pi}{2})$, $g(\pm\frac{\pi}{2\lambda})=0$



Consider $h(v)=\int_0^v\sqrt{1-A^2\lambda^2\sin^2(\lambda t)}\,dt$.

We distinguish 3 cases

(1) $\underline{\lambda^2A^2>1}$ Then $h(v)$ is only defined if $|\sin(\lambda v)|<\lambda A$. But then $g(v)>0$, and we do not get a compact surface of rotation

(2) $\underline{\lambda^2A^2<1}$ Then $h(v)$ and $g(v)$ defined for all v, but $g(v)$ is sometimes 0, and we don't get a smooth surface

(3) $\underline{A\lambda=1}$ Then $g(v)=\frac{1}{\lambda}\cos(\lambda v)$, $h(v)=\frac{1}{\lambda}\sin\lambda v$, and we get a sphere of radius $\frac{1}{\lambda}$