

5, 6, 12

Recall that γ_v^p is unique geodesic (1)

s.t. $\gamma_v^p(0) = p, \quad \gamma_v^p'(0) = v.$

$$\exp_p(v) = \gamma_v^p(1)$$

$$\exp_{f(p)}(df_p(v)) = \gamma_{df_p(v)}^{f(p)}(1)$$

f isometry $\Rightarrow f(\text{geodesic}) = \text{geodesic}$.

Therefore, $f \circ \gamma_v^p = \text{geodesic } \beta$ s.t.

$$\beta(0) = f(\gamma_v^p(0)) = f(p)$$

$$\beta'(0) = (f \circ \gamma_v^p)'(0) = df_p(\gamma_v^p'(0)) = df_p(v)$$

$$\Rightarrow \beta = \gamma_{df_p(v)}^{f(p)}$$

5, 7, 1

$$A_p = \int_{0 \leq r \leq p} \int_{0 \leq \theta \leq 2\pi} \sqrt{EG - F^2} \, dr \, d\theta \quad E=1, G=h^2, F=0$$

$$= \int_{r, \theta} h \, dr \, d\theta = \int_{r, \theta} \left(r - \frac{K(p)}{6} r^3 + \dots \right) dr \, d\theta$$

$$= 2\pi \left(\frac{p^2}{2} - \frac{K(p)}{24} p^4 \right) + \dots \rightarrow 0$$

$$\Rightarrow \frac{\pi}{12} K(p) = \frac{\pi p^2 - A_p}{p^4} + \dots \rightarrow 0$$

$$K(p) = 12 \lim_{p \rightarrow 0} \frac{\pi p^2 - A_p}{\pi p^4}$$

(2)

Proof that any point p has a nbhd V
 s.t. if $q \in V$, the shortest curve from
 p to q is a unique geodesic.

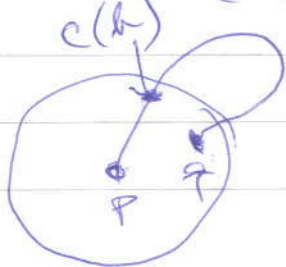
Let $V = \exp_p(B_p(\epsilon))$, where $\exp_p|_{B_p(\epsilon)}$ is
 a diffeomorphism so $q \in V$ can then
 be written $q = \exp_p(v(\theta_0))$, where
 $v(\theta)$ parametrizes the unit circle in $T_p S$.
 $s \mapsto \gamma(s) = \exp_p(sv(\theta_0))$ is then geodesic from
 p to q , and $l(\gamma) = \rho < \epsilon$.

Let $c: [0, a] \rightarrow S$ be another curve in S from
 p to q . Then $l(c) = \int_0^a |c'(t)| dt$.

(i) If $c[0, a] \subset V$, we can write $c(t) = x(r(t), \theta(t))$,
 where $x(r, \theta)$ is parametrization by geodesic
 polar coordinates. Then $ds^2 = dr^2 + h^2 d\theta^2$,
 and we have
 $|c'(t)|^2 = (r'(t))^2 + h^2(r(t), \theta(t)) \theta'(t)^2$.

Hence $l(c) \geq \int_0^a r'(t) dt = r(a) = \rho$,
 with equality only if $\theta(t)$ is constant $= \theta_0$.

(ii) If $c[0, a] \not\subset V$, there must be a $b \in [0, a)$
 s.t. $\epsilon > r(b) > \rho$. But then $l(c) > l(c[0, b])$
 and $l(c[0, b]) > \rho$ by the same argument.



and $c([0, b]) \subset V$