(1)

5,6,12 Recall that γ_v^p is unique geodesic s.t. $\gamma_v^p(0)=0$, $\gamma_v^{p\,\prime}(0)=v$.

$$\exp_p(v)=\gamma_v^p(1)$$

$$\exp_{f(p)}(df_p(v))=\gamma_{df_p(v)}^{f(p)}(1)$$

f isometry $\Rightarrow$ f(geodesic) = geodesic.

Therefore, $f\circ\gamma_v^p$ = geodesic β s.t.

$$\beta(0)=f(\gamma_v^p(0))=f(p)$$

$$\beta'(0)=(f\circ\gamma_v^p)'(0)=df_p(\gamma_v^{p\,\prime}(0))=df_p(v)$$

$$\Rightarrow \underline{\beta=\gamma_{df_p(v)}^{f(p)}}$$

5,7,1 $$A_\rho=\iint\limits_{\substack{0\le r\le\rho\\0\le\theta\le2\pi}}\sqrt{EG-F^2}\,dr\,d\theta \qquad E=1,\ G=h^2,\ F=0$$

$$=\iint\limits_{r,\theta}h\,dr\,d\theta=\iint\limits_{r,\theta}\left(r-\frac{K(p)}{6}r^3+\cdots\right)dr\,d\theta$$

$$=2\pi\left(\frac{\rho^2}{2}-\frac{K(p)}{24}\rho^4\right)+(\cdots)\rho^4, \quad (\cdots)\to 0$$

$$\Rightarrow \frac{\pi}{12}K(p)=\frac{\pi\rho^2-A_\rho}{\rho^4}+(\cdots), \quad (\cdots)\to 0$$

$$K(p)=12\lim_{\rho\to0}\frac{\pi\rho^2-A_\rho}{\pi\rho^4}$$

Proof that any point p has a nbhd V s.t. if $q \in V$, the shortest curve from p to q is a unique geodesic.

Let $V = \exp_p(B_p(\varepsilon))$, where $\exp_p | B_p(\varepsilon)$ is a diffeomorphism. $q \in V$ can then be written $q = \exp_p(\rho v(\theta_0))$, where $v(\theta)$ parametrizes the unit circle in T_pS. $s \mapsto \gamma(s) = \exp_p(s v(\theta_0))$ is then geodesic from p to q, and $l(\gamma) = \rho < \varepsilon$.

Let $c : [0,a] \to S$ be another curve in S from p to q. Then $l(c) = \int_0^a |c'(t)|\,dt$.

(i) If $c[0,a] \subset V$, we can write $c(t) = X(r(t), \theta(t))$, where $X(r,\theta)$ is paramtrization by geodesic polar coordinates. Then $ds^2 = dr^2 + h^2 d\theta^2$, and we have

$$|c'(t)|^2 = (r'(t))^2 + h^2(r,\theta)\,\theta'(t)^2 .$$

Hence $l(c) \geq \int_0^a r'(t)\,dt = r(a) = \rho$, with equality only if $\theta(t)$ is constant $= \theta_0$

(ii) If $c[0,a] \not\subset V$, there must be a $b \in [0,a)$ s.t. $\varepsilon > r(b) > \rho$. But then $l(c) > l(c|[0,b])$ and $l(c[0,b]) > \rho$ by the same argument,

and $c([0,b]) \subset V$