MANDATORY ASSIGNMENT FOR MAT4510 FALL 2015

Turn in your written answers before Thursday October 22nd at 14:30, in the assignment box on the 7th floor of Niels Henrik Abel's house. There are often long queues just before the deadlines, so you are adviced to turn in your paper early. Remember to use the official front page. If you need a delayed deadline, due to illness or other circumstances, you must apply for an extension to Helena Båserud Mathisen (room B718, NHA, e-mail: studieinfo@math.uio.no, phone 22 85 59 07). Remember that illness has to be documented by a medical doctor. See http://www.mn.uio.no/math/english/studies/admin/mandatory-assignments/index.html for more information about rules concerning compulsory assignments at the Department of Mathematics. The assignment is mandatory, and students who do not get their paper accepted will not get access to the final exam. To get this paper accepted you must answer at least 50% of the eight parts, 1(a-d) and 2(a-d), of the problem set correctly. You may get partial credit for partial solutions, so turn in all your work.

Problem 1

(a) Watch "Möbius Transformations Revealed" by Douglas N. Arnold and Jonathan Rogness (my fourth cousin, not me), on YouTube

`http://www.youtube.com/watch?v=JX3VmDgiFnY`

or at

`http://www.ima.umn.edu/~arnold/moebius/` .

Then read the article [AR08]

`http://www.ima.umn.edu/~arnold/papers/moebius.pdf` .

Who composed the music played in the video?

(b) Consider any fractional linear transformation (FLT)

$$m(z) = \frac{az+b}{cz+d}$$

with a, b, c and $d \in \mathbb{R}$ and $ad - bc = 1$, preserving the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Prove that m can be expressed as a composition of fractional linear transformations, each preserving $\mathbb{H}$, of the following three kinds:

(1) $m_1(z) = \alpha z$ with $\alpha > 0$ real (of hyperbolic type),
(2) $m_2(z) = z + \beta$ with β real (of parabolic type), and
(3) $m_3(z) = -1/z$ (of elliptic type).

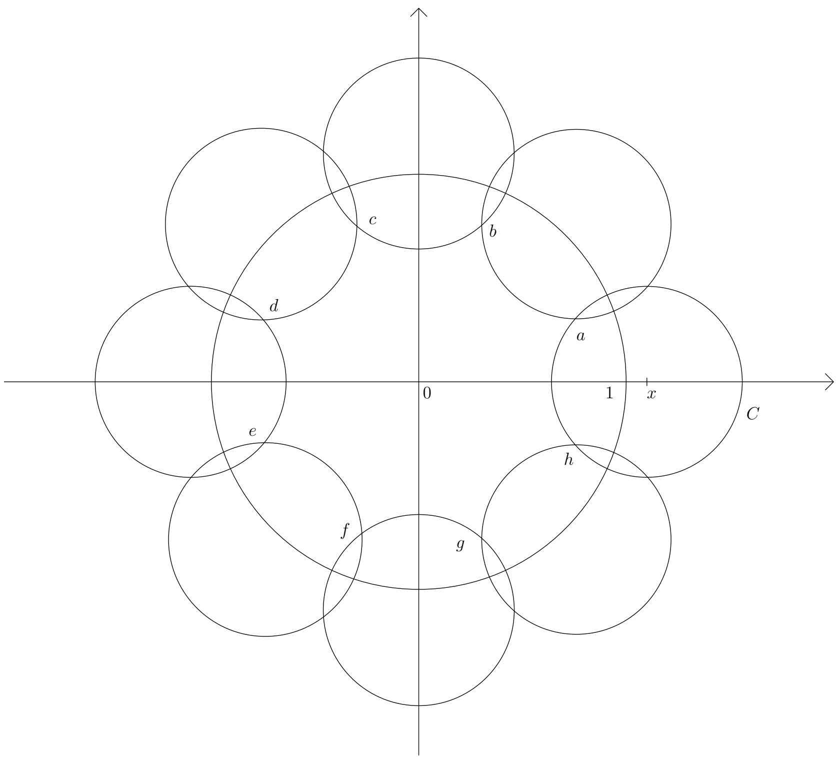
(c) Theorem 1 on page 1228 of [AR08] asserts that every holomorphic Möbius transformation $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ can be obtained as a composition

$$P_{S'} \circ T \circ P_S^{-1} ,$$

where $P_{S'} \colon S' \to \bar{\mathbb{C}}$ is a generalized stereographic projection, $P_S^{-1} \colon \bar{\mathbb{C}} \to S$ is a generalized inverse stereographic projection, and $T \colon S \to S'$ is a rigid motion of $\mathbb{R}^3$. Explain how each of the three FLTs m_1, m_2 and m_3 from part (b) can be realized in this way. (Hint: The answers for m_1 and m_2 are given in [AR08]. Be careful when you answer the case of m_3.)

(d) There is an analogue of Theorem 1 of [AR08], stating that each FLT preserving $\mathbb{H}$ can be obtained as a composition $P_{S'} \circ T \circ P_S^{-1}$, for a more restricted class of admissible spheres S and a more restricted class of rigid motions T. Determine these restricted classes of admissible spheres and rigid motions.

Date: September 23rd 2015.



Problem 2

(a) Let $w = \cos(\pi/8) + i\sin(\pi/8)$ be a primitive 16-th root of unity, and let $0 < r < 1$. Consider a regular hyperbolic octagon $abcdefgh$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, with vertices at $a = rw$, $b = rw^3$, $c = rw^5$, $d = rw^7$, $e = rw^9$, $f = rw^{11}$, $g = rw^{13}$ and $h = rw^{15} = r\bar{w}$. The edge $[h, a]$ is the hyperbolic line segment from $r\bar{w}$ to rw, which is part of a Euclidean circle C meeting the unit circle $S^1 = \partial\mathbb{D}$ at right angles. The Euclidean center of C is a point $x \in \mathbb{R} \subset \mathbb{C}$ on the real axis, with $x > 1$. Prove that the Euclidean radius of C is $\sqrt{x^2 - 1}$.

(b) We add the condition (on r) that each interior angle in the octagon is equal to $\pi/4 = 45°$, so that the sum $\angle hab + \angle abc + \cdots + \angle gha$ of the interior angles is equal to $2\pi = 360°$. Let L be the Euclidean tangent line to C at $a = rw$, intersecting the real axis at $y > 0$. Explain why $\angle y0a = \pi/8 = 22.5°$, $\angle ya0 = \pi/8 = 22.5°$, $\angle xay = \pi/2 = 90°$ and $\angle 0xa = \angle yxa = \pi/4 = 45°$.

(c) Prove that

$$x = \sqrt{\frac{\sqrt{2}+1}{2}} \quad \text{and} \quad r = \frac{1}{\sqrt[4]{2}}.$$

(Hint: Show that $x = \sqrt{x^2-1} + \sqrt{2}\sqrt{x^2-1}$ to determine x, and establish that $2\cos(\pi/8) = \sqrt{\sqrt{2}+2}$ on the way to calculating r.)

(d) Explain how to compute the hyperbolic area and the hyperbolic circumference of the octagon $abcdefgh$. (Give an exact expression in terms of π in the case of the area, and a numerical approximation to two decimal places in the case of the circumference.)