Section 5.1

Exercise 4. Let $\beta(u) = (f(u), g(u))$ be an embedded curve such that $f(u) > 0$ for all u. The surface of revolution obtained by rotating β around the z-axis has local parametrization

$$x(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where u varies as for β, and $v \in J$ for $J = (-\pi, \pi)$ and $J = (0, 2\pi)$.

Then

$$\begin{aligned} x_u &= x_u(u, v) = (f'(u)\cos v, f'(u)\sin v, g'(u)) \\ x_v &= x_v(u, v) = (-f(u)\sin v, f(u)\cos v, 0) \end{aligned}$$

so

$$x_u \times x_v = (-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u))$$

has length

$$\begin{aligned} \|x_u \times x_v\| &= \sqrt{f(u)^2 g'(u)^2(\cos^2 v + \sin^2 v) + f(u)^2 f'(u)^2} \\ &= f(u)\sqrt{f'(u)^2 + g'(u)^2}\,. \end{aligned}$$

The parametrization x is regular if $\|x_u \times x_v\| \neq 0$, or equivalently, if $\beta'(u) = (f'(u), g'(u)) \neq 0$, i.e., when β is regular.

Exercise 5. (TBW)

Exercise 6. (TBW)

Section 5.3

Exercise 1. The first fundamental form is

$$\begin{aligned} E &= x_u \cdot x_u = f'(u)^2(\cos^2 v + \sin^2 v) + g'(u)^2 \\ &= f'(u)^2 + g'(u)^2\,, \\ F &= x_u \cdot x_v = f(u)f'(u)(-\cos v \sin v + \sin v \cos v) + 0 \\ &= 0 \\ G &= x_v \cdot x_v = f(u)^2(\sin^2 v + \cos^2 v) + 0 \\ &= f(u)^2\,. \end{aligned}$$

Exercise 2. (TBW)

Section 5.4

Exercise 5. (TBW)

Section 5.5

Exercise 1. The unit normal vector at $p = x(u, v)$ is

$$\begin{aligned} N_p &= \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{(-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u))}{f(u)\sqrt{f'(u)^2 + g'(u)^2}} \\ &= \frac{(-g'(u)\cos v, -g'(u)\sin v, f'(u))}{\sqrt{f'(u)^2 + g'(u)^2}} \end{aligned}$$

The second order partial derivatives are

$$\begin{aligned} x_{uu} &= (f''(u)\cos v, f''(u)\sin v, g''(u)) \\ x_{uv} &= (-f'(u)\sin v, f'(u)\cos v, 0) \\ x_{vv} &= (-f(u)\cos v, -f(u)\sin v, 0)\,, \end{aligned}$$

so the second fundamental form is

$$\begin{aligned} e = N \cdot x_{uu} &= \frac{-f''(u)g'(u)\cos^2 v - f''(u)g'(u)\sin^2 v + f'(u)g''(u)}{\sqrt{f'(u)^2 + g'(u)^2}} \\ &= \frac{f'(u)g''(u) - f''(u)g'(u)}{\sqrt{f'(u)^2 + g'(u)^2}} \\ f = N \cdot x_{uv} &= \frac{f'(u)g'(u)\sin v\cos v - f'(u)g'(u)\sin v\cos v + 0}{\sqrt{f'(u)^2 + g'(u)^2}} \\ &= 0 \\ g = N \cdot x_{vv} &= \frac{f(u)g'(u)\cos^2 v + f(u)g'(u)\sin^2 v + 0}{\sqrt{f'(u)^2 + g'(u)^2}} \\ &= \frac{f(u)g'(u)}{\sqrt{f'(u)^2 + g'(u)^2}}\,. \end{aligned}$$

Hence

$$EG - F^2 = f(u)^2(f'(u)^2 + g'(u)^2)$$

and

$$eg - f^2 = \frac{(f'(u)g''(u) - f''(u)g'(u))f(u)g'(u)}{f'(u)^2 + g'(u)^2}\,,$$

so the Gaussian curvature at p is

$$K(p) = \frac{eg - f^2}{EG - F^2} = \frac{(f'(u)g''(u) - f''(u)g'(u))g'(u)}{(f'(u)^2 + g'(u)^2)^2 f(u)}\,.$$

More concisely,

$$K = \frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f}\,.$$

If $K = 0$ then $f'g'' - f''g' = 0$ or $g' = 0$. In the latter case, $g = C$ is constant, so the surface is contained in the plane $z = C$. If $f' = 0$ then $f'' = 0$ and $f = C$ is constant, so the surface is contained in the cylinder $x^2 + y^2 = C^2$. If $f' \neq 0$ and $g' \neq 0$ then $f''/f' = g''/g'$, so $\ln|f'| = \ln|g'| + C$. Hence $f' = \pm e^C g' = Ag'$ and $f = Ag + B = A(g + B/A)$ for some constants A and B. In this case the surface is contained in the cone $x^2 + y^2 = A^2(z + B/A)^2$.

Exercise 2. Let $0 < a < b$. The torus is the surface of revolution of the curve $\beta(u) = (f(u), g(u))$ with

$$\begin{aligned} f(u) &= a\cos u + b \\ g(u) &= a\sin u\,. \end{aligned}$$

Here

$$\begin{aligned} f'(u) &= -a\sin u \\ f''(u) &= -a\cos u \\ g'(u) &= a\cos u \\ g''(u) &= -a\sin u \end{aligned}$$

so

$$K = \frac{(a^2 \sin^2 u + a^2 \cos^2 u) a \cos u}{(a^2 \sin^2 u + a^2 \cos^2 u)^2 (a \cos u + b)} = \frac{\cos u}{a(a \cos u + b)} .$$

Here $a \cos u + b \geq -a + b > 0$ for all u, so K has the same sign as $\cos u$, i.e., $K > 0$ if $u \in (-\pi/2, \pi/2)$ modulo 2π, and $K < 0$ if $u \in (\pi/2, 3\pi/2)$ modulo 2π. The curvature is zero when $u = \pm\pi/2$ modulo 2π, i.e., at the top and the bottom of the torus (where $z = \pm a$).

Exercise 3. Suppose that $\beta = \alpha$ is parametrized by arc length, so that $f'(u)^2 + g'(u)^2 = 1$. Differentiating $(f')^2 + (g')^2 = 1$ we get $2f'f'' + 2g'g'' = 0$ and $g'g'' = -f'f''$. Hence

$$(f'g'' - f''g')g' = f'g'g'' - f''(g')^2 = -(f')^2 f'' - f''(1 - (f')^2) = -f'' ,$$

so that

$$K = \frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f} = \frac{-f''}{f} .$$

If K is constant, we get $f'' = -Kf$. If $K = k^2 > 0$ then

$$f(u) = A \cos(k(u - u_0))$$

for some constants A and u_0. Then $f'(u) = -Ak \sin(k(u - u_0))$, so

$$g'(u) = \pm\sqrt{1 - A^2 K \sin^2(k(u - u_0))} .$$

Hence g is an indefinite integral of the right hand side, with respect to u, which in general is an elliptic integral (of the second kind).

To get a closed surface, as $f(u) \to 0$ we must have $g'(u) \to 0$, which implies $A^2 K = 1$, i.e., $A = 1/k$. Then $g'(u) = \pm \cos(k(u - u_0))$ and $g(u) = (1/k) \sin(k(u - u_0)) + C$. Reparametrizing with $u_0 = 0$, and possibly replacing u with $-u$, we conclude that

$$\alpha(u) = (\frac{\cos(ku)}{k}, \frac{\sin(ku)}{k} + C)$$

parametrizes the circle with radius $1/k = 1/\sqrt{K}$ centered at $(x, y, z) = (0, 0, C)$, and the surface of revolution is the sphere with radius $1/k$ centered at the same point.

If $K = 0$ then

$$f(u) = Au + B$$

for some constants A and B, so $f'(u) = A$, $g'(u) = \pm\sqrt{1 - A^2}$ and $g(u) = \pm\sqrt{1 - A^2} \cdot u + C$. Hence

$$\alpha(u) = (B, C) + (A, \pm\sqrt{1 - A^2}) \cdot u$$

is a straight line. Hence the surface of revolution lines on a plane, a cylinder or a cone, and is never closed.

If $K = -k^2 < 0$ then

$$f(u) = Ae^{ku} + Be^{-ku}$$

for $A, B \geq 0$ (not both zero). Then $f'(u) = Ake^{ku} - Bke^{-ku}$ and

$$g'(u) = \pm\sqrt{1 - (Ake^{ku} - Bke^{-ku})^2} .$$

Again $g(u)$ is given by an elliptic integral. These surfaces are never closed, since $f(u) \to \infty$ as $u \to \pm\infty$.

Exercise 4. (a) Let S_1 be the surface of revolution obtained from the regular curve

$$\beta(u) = (u, \ln u)$$

for $u > 0$, hence parametrized by

$$x(u, v) = (u \cos v, u \sin v, \ln u) .$$

Here $f(u) = u$ and $g(u) = \ln u$, so $f'(u) = 1$, $f''(u) = 0$, $g'(u) = 1/u$ and $g''(u) = -1/u^2$. Hence

$$K = \frac{(-1/u^2)(1/u)}{(1 + 1/u^2)^2 u} = \frac{-1}{(1 + u^2)^2}$$

is always negative. (Omitting the step-by-step method.)

(b) Let S_2 be the helicoid (a ruled surface) parametrized by

$$y(u, v) = (u \cos v, u \sin v, v)$$

for $u > 0$ and all v. We calculate

$$\begin{aligned}
y_u &= (\cos v, \sin v, 0) \\
y_v &= (-u \sin v, u \cos v, 1) \\
E &= 1 \\
F &= 0 \\
G &= 1 + u^2 \\
y_u \times y_v &= (\sin v, - \cos v, u) \\
\|y_u \times y_v\| &= \sqrt{1 + u^2} \\
N &= \frac{(\sin v, - \cos v, u)}{\sqrt{1 + u^2}} \\
y_{uu} &= (0, 0, 0) \\
y_{uv} &= (- \sin v, \cos v, 0) \\
y_{vv} &= (-u \cos v, -u \sin v, 0) \\
e &= 0 \\
f &= \frac{-1}{\sqrt{1 + u^2}} \\
g &= 0
\end{aligned}$$

so that

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-1/(1 + u^2)}{1 + u^2} = \frac{-1}{(1 + u^2)^2} .$$

(c) A smooth map $f \colon S_2 \to S_1$ maps $p = y(u, v)$ to a point $q = f(p) = x(u_1, v_1)$, where $u_1 > 0$ and v_1 depend smoothly on $u > 0$ and v. For $K(p) = -1/(1 + u^2)^2$ in S_2 to be equal to $K(q) = -1/(1 + u_1^2)^2$ in S_1 we must have $u_1 = u$. Hence for any smooth function $v_1 = h(u, v)$ the map f taking $y(u, v)$ to $x(u, h(u, v))$ will preserve the Gaussian curvature. For f to be a local isometry, it must take the first fundamental form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 + u^2 \end{bmatrix}$$

of S_2 to the first fundamental form

$$\begin{bmatrix} f'(u)^2 + g'(u)^2 & 0 \\ 0 & f(u)^2 \end{bmatrix} = \begin{bmatrix} 1 + 1/u^2 & 0 \\ 0 & u^2 \end{bmatrix}$$

of S_1, in the sense of Exercise 5.3.3. Here

$$x^{-1} f y \colon (u, v) \mapsto (u, h(u, v))$$

has Jacobian

$$J(x^{-1}fy) = \begin{bmatrix} 1 & 0 \\ h_u & h_v \end{bmatrix} ,$$

so the condition is that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1+u^2 \end{bmatrix} = \begin{bmatrix} 1 & h_u \\ 0 & h_v \end{bmatrix} \begin{bmatrix} 1+1/u^2 & 0 \\ 0 & u^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ h_u & h_v \end{bmatrix} .$$

There are no such functions h_u and h_v, as can be seen by multiplying out.