PROPOSED SOLUTION TO THE MANDATORY ASSIGNMENT FOR MAT4510 FALL 2015

Problem 1

(a) Robert Schumann.

(b) If $c = 0$ then $ad = 1$, so $m(z) = (az+b)/(0z+d) = a^2z+ab$ is the composite $m = m_2 \circ m_1$ of $m_1(z) = a^2 z$ with $a^2 > 0$ followed by $m_2(z) = z + ab$.

If $c \neq 0$ then $m(z) = (az+b)/(cz+d) = (a/c) - 1/(c^2z+cd)$ is the composite $m = m_2' \circ m_3 \circ m_2 \circ m_1$ of $m_1(z) = c^2 z$ with $c^2 > 0$, $m_2(z) = z + cd$, $m_3(z) = -1/z$ and $m_2'(z) = z + (a/c)$.

(c) In each case we can take S to be the unit sphere S^2 in $\mathbb{R}^3$. For m_1 let $T(x, y, z) = (x, y, z + \alpha - 1)$. For m_2 let $T(x, y, z) = (x + \beta, y, z)$. For m_3 let $T(x, y, z) = (-x, y, -z)$ be reflection (= rotation by π radians) in the y-axis. In each case let $S' = T(S)$.

(d) The FLTs preserving $\mathbb{H}$ are realized as compositions $P_{S'} \circ T \circ P_S^{-1}$ where S and S' are admissible spheres with center in the xz-plane (i.e., with $y = 0$), and T is a rigid motion preserving that plane.

Problem 2

(a) The unit circle S^1, with radius 1, and the circle C, with radius R, meet at right angles at a point p. By the Pythagorean theorem for the triangle $\triangle 0px$ we have $1^2 + R^2 = x^2$, so $R = \sqrt{x^2 - 1}$.

(b) $\angle y0a = \angle 10w = \pi/8$ by the definition of w. The Euclidean angle $\angle ya0$ equals the hyperbolic angle $\angle ha0$, which by symmetry is half the hyperbolic angle $\angle hab = \pi/4$, hence $\angle ya0 = \pi/8$. The radius $[x, a]$ is orthogonal to the tangent line L of C at a, hence also to the segment $[a, y]$, so $\angle xay = \pi/2$. Thus $\angle 0ya = \pi - (\angle y0a + \angle ya0) = \pi - (\pi/8 + \pi/8) = 3\pi/4$, so $\angle ayx = \pi - \angle 0ya = \pi - 3\pi/4 = \pi/4$, and $\angle yxa = \pi - (\angle xay + \angle ayx) = \pi - (\pi/2 + \pi/4) = \pi/4$.

(c) The Euclidean length y of $[0, y]$ equals the length of $[y, a]$, since $\triangle 0ya$ is isosceles with $\angle y0a = \pi/8 = \angle ya0$. Likewise the length of $[y, a]$ equals the length R of $[x, a]$, since $\triangle axy$ is right isosceles with $\angle ayx = \pi/4 = \angle yxa$. Hence $y = R = \sqrt{x^2 - 1}$.

By Pythagoras for $\triangle axy$ the length of $[y, x]$ is $\sqrt{2}R = \sqrt{2}\sqrt{x^2-1}$. Splitting $[0, x]$ at y we get $x = y + \sqrt{2}R = (1 + \sqrt{2})\sqrt{x^2-1}$. Hence $x^2 = (1+\sqrt{2})^2(x^2-1)$, so $(1+\sqrt{2})^2 = 3+2\sqrt{2} = (2+2\sqrt{2})x^2 = 2(1+\sqrt{2})x^2$ and $x^2 = (1+\sqrt{2})/2$, giving $x = \sqrt{(\sqrt{2}+1)/2}$.

Splitting $\triangle 0ya$ into two right triangles we find $r = 2\cos(\pi/8)y$, hence $r^2 = 4\cos^2(\pi/8)y^2$. From $\cos(2t) = 2\cos^2 t - 1$ we get $4\cos^2 t = 2(\cos(2t)+1)$, so $4\cos^2(\pi/8) = 2(\sqrt{2}/2+1) = \sqrt{2}+2$. Furthermore, $y^2 = x^2 - 1 = (\sqrt{2}+1)/2 - 1 = (\sqrt{2}-1)/2$. Hence $r^2 = (\sqrt{2}+2)(\sqrt{2}-1)/2 = \sqrt{2}/2$ and $r = \sqrt{\sqrt{2}/2} = 1/\sqrt[4]{2}$.

(d) The hyperbolic area of the octagon is 8 times the area of the hyperbolic triangle $\triangle 0ha$, with interior angles $2\pi/8 = \pi/4$ at 0, $\pi/8$ at h and $\pi/8$ at a. Hence $\triangle 0ha$ has area $\pi - (\pi/4 + \pi/8 + \pi/8) = \pi/2$, and the area of the octagon is $8 \cdot \pi/2 = 4\pi$.

The hyperbolic circumference is 8 times the length $d_{\mathbb{D}}(h, a)$ of the hyperbolic segment $[h, a]$, which satisfies

$$\cosh(d_{\mathbb{D}}(h, a))) = 1 + \frac{2|a-h|^2}{(1-|h|^2)(1-|a|^2)} = 1 + \frac{2(2r\sin(\pi/8))^2}{(1-r^2)(1-r^2)} = 5 + 4\sqrt{2}$$

so $d_{\mathbb{D}}(h, a) = \cosh^{-1}(5 + 4\sqrt{2}) = 3.057\ldots$ and the circumference is $8\cosh^{-1}(5 + 4\sqrt{2}) = 24.457\ldots \approx 24.46$.

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