UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Examination in: MAT4510 — Goeometric Structures

Day of examination: Monday Dec 17. 2018.

Examination hours: $14.30 - 18.30$

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Christoffel

All problems (1a, 1b, 2a, 2b etc.) count for 10 points each. You have to explain all answers, and show enough details so that it is easy to follow your arguments. At the end of this document you will find some definitions that might be handy. You may answer the exam in either English or Norwegian.

Problem 1

Let $f \in \text{Möb}^+(\mathbb{H})$ be the map

$$f(z) = \frac{z-1}{z+1}.$$

Determine whether f is hyperbolic, parabolic or elliptic.

Problem 2

Let $S \subset \mathbb{R}^3$ be a smooth regular surface. Explain what the Gauss map $N : S \to S^2$ is, and use this to define Gaussian curvature.

Problem 3

Let $S \subset \mathbb{R}^3$ be a smooth regular surface, and let $\gamma : [0,1] \to S$ be a smooth curve. Explain geometrically what the covariant second order derivative of γ is, and explain what it would mean that γ is a geodesic.

Problem 4

Let $g(u) = u, h(u) = e^{-u}$, and let

$$x(u, v) = (g(u)\cos(v), g(u)\sin(v), h(u)),$$

for $(u, v) \in (0, \infty) \times [0, 2\pi]$, parametrize the surface S of revolution around the z-axis in $\mathbb{R}^3$ with coordinates (x, y, z).

(a) Compute the first fundamental form.

(b) Compute the Gaussian curvature.

(c) Prove that for a fixed v_0, the curves $x(u, v_0)$ are geodesics.

(d) Prove that for a fixed u_0, the curves $x(u_0, v)$ are not geodesics.

(e) Use the Gauss-Bonnet Theorem to prove that for any $0 < r < R < \infty$, the Euler characteristic of the annulus $A = x([r, R] \times [0, 2\pi])$ is zero.

Solution:

(a) We have that

$$x_u = (\cos(v), \sin(v), -e^{-u}),$$

and that

$$x_v = (-u\sin(v), u\cos(v), 0).$$

So we get that

$$E = x_u \cdot x_u = \cos^2(v) + \sin^2(v) + e^{-2u} = 1 + e^{-2u},$$

$$F = x_u \cdot x_v = \cos(v) \cdot (-u\sin(v)) + \sin(v) \cdot u\cos(v) + 0 = 0,$$

and

$$G = x_v \cdot x_v = u^2 \sin^v + u^2 \cos^2 v = u^2.$$

So the first fundamental form is

$$\mathbf{ds^2 = (1 + e^{-2u})du^2 + u^2 dv^2.}$$

(b) Here, the quickest is to remember the general formula for a surface of revolution

$$K(x(u, v)) = \frac{h'(u)(g'(u)h''(u) - h'(u)g''(u))}{g(u)(g'(u)^2 + h'(u)^2)^2}.$$

We have that $h'(u) = -e^{-u}, g(u) = u, g'(u) = 1, g''(u) = 0, h''(u) = e^{-u}$, and so

$$K(x(u, v)) = \frac{-e^{-u}(1 \cdot e^{-u} - 0)}{u(1 + e^{-2u})^2} = \frac{\mathbf{-e^{-2u}}}{\mathbf{u(1 + e^{-2u})^2}}.$$

Alternatively, one can calculate the coefficients of the second fundamental form e, f and g, and use the formula

$$K = \frac{ef - g^2}{EF - G^2}.$$

(c) First we calculate the Christoffel symbols with respect to the parametrisation, using the fact at the end of the problem sheet. Here $E = 1 + e^{-2u}, F = 0$ and $G = u^2$, so we see that

$$\begin{bmatrix} 1 + e^{-2u} & 0 \\ 0 & u^2 \end{bmatrix} \quad \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} -e^{-2u} & 0 & -u \\ 0 & u & 0 \end{bmatrix}$$

So we get that for a curve $\alpha(t) = (u(t), v(t))$ we have that

$$D\alpha''(t) = (u''(t) + u'(t)^2 \Gamma_{11}^1 + v'(t)^2 \Gamma_{22}^1)x_u$$
$$+ (v''(t) + u'(t)v'(t)\Gamma_{12}^2)x_v.$$

For the curve $\alpha(t) = (t, v_0)$ we see that $v''(t) = v'(t) = 0$, so $D\alpha''(t)$ is indeed always a multiple of $\alpha'(t) = x_u$.

(d) If we consider the curve $\alpha(t) = (u_1, t)$ (which is even a constant speed curve), we see from (c) that the x_u-component of $D\alpha''(t)$ is $\Gamma_{22}^1 \cdot x_u$ which is never zero, so $D\alpha''(t)$ is never a multiple of $\alpha'(t) = x_v$.

(e) If we let α_r denote the curve $u = r$ and if we let α_R denote the curve $u = R$, the annular region $A \subset S$ is bounded by the smooth curves $x(\alpha_r)$ and $x(\alpha_R)$. We start by calculating the integral

$$\int_{\partial A} k_g ds,$$

and so first we parametrise α_r and α_R by arc length. We get $\alpha_r(t) = (r, t/r)$ and $\alpha_R(t) = (R, t/R)$. We get that

$$D\alpha_r''(t) = v'(t)^2 \Gamma_{22}^1 x_u = (1/r)^2 (-r)(1 + e^{-2r})^{-1} x_u = \frac{-1}{r(1 + e^{-2r})} x_u,$$

and similarly

$$D\alpha_R''(t) = \frac{-1}{R(1 + e^{-2R})} x_u.$$

Now along α_r the inward pointing unit normal is $\frac{x_u}{\sqrt{1+e^{-2r}}}$ and so k_g^r becomes $\frac{-1}{r(1+e^{-2r})^{1/2}}$, and if we consider α_R the normal points the other way and k_g^R becomes $\frac{1}{R(1+e^{-2R})^{1/2}}$. It follows that

$$\int_{\partial A} k_g ds = 2\pi \left(\frac{1}{(1 + e^{-2R})^{1/2}} - \frac{1}{(1 + e^{-2r})^{1/2}} \right).$$

Next we consider the integral

$$\int\int_A K dA.$$

This is calculated in local coordinates

$$\int\int_A K dA = \int_0^{2\pi} \int_r^R \frac{-e^{-2u}}{u(1 + e^{-2u})^2} \sqrt{u^2(1 + e^{-2u})} \, du \, dv$$

$$= \int_0^{2\pi} \int_r^R \frac{-e^{-2u}}{(1 + e^{-2u})^{3/2}} \, du \, dv$$

$$= 2\pi [-(1 + e^{-2u})^{-1/2}]_r^R$$

$$= -\int_{\partial A} k_g ds.$$

The Gauss-Bonnet theorem states that

$$\int\int_A K dA + \int_{\partial A} k_g ds + \sum_k \epsilon_k = \chi(A),$$

where the ϵ_k's are the turning angles, and $\chi(A)$ is the Euler characteristic. In this case ∂A is smooth, and so there are no turning angles, and it follows that $\chi(A) = 0$.

Problem 5

(a) State the classification theorem for compact Riemannian surfaces.

(b) You may now take for granted that any compact Riemannian surface admits a Riemannian metric with curvature constantly equal to -1, 0, or 1. For each of the surfaces in the classification in (a), determine the corresponding constant curvature.

THE END

Some facts:

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \quad \begin{bmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{bmatrix} = \begin{bmatrix} E_u/2 & E_v/2 & F_v - G_u/2 \\ F_u - E_v/2 & G_u/2 & G_v/2 \end{bmatrix}$$