Final exam Mat4510, 2014 – suggestions for solutions

Problem 1

1a) $f(z) = \dfrac{az+b}{cz+d}$ must satisfy either $(1)\colon\ f(0) = -1,\ f(\infty) = 1$, or $(2)\colon\ f(0) = 1,\ f(\infty) = -1$.

Case (1) gives $b = -d \neq 0$ and $a = c \neq 0$, or $f(z) = \dfrac{z + \frac{b}{a}}{z + (-\frac{b}{a})}$. Since the determinant must be positive, $\frac{b}{a} < 0$. Setting $c = -\frac{b}{a}$ gives the first type. A similar argument in case (2) gives the other. Alternatively, precompose case (1) with $h(z) = -1/z$, which interchanges 0 and ∞.

On the other hand, a simple calculation shows that both the two forms of $f(z)$ satisfy $|f(it)| = 1$, for $t \in \mathbb{R}$.

1b) Let $f(z) = -\dfrac{z-c}{z+c}$ where $c < 0$. Then the fixpoints of f are $z = \frac{1}{2}(-1-c\pm\sqrt{c^2+6c+1})$. The solutions of $c^2+6c+1 = 0$ are $c = -3\pm2\sqrt{2}$, and both are negative real numbers. Hence the classification is as follows:

- f *parabolic*: One real fixpoint, when $c = -3\pm2\sqrt{2}$.
- f *hyperbolic*: Two real fixpoints. This happens when $c^2+6c+1 > 0$, i.e. $c \in (-\infty, -3-2\sqrt{2}) \cup (-3+2\sqrt{2}, 0)$
- f *elliptic*: Two conjugate complex fixpoints, one in $\mathbb{H}$; when $c \in (-3-2\sqrt{2}, -3+\sqrt{2})$.

Assume $f(i) = i$, i.e. $-\dfrac{i-c}{i+c} = i$; $c = -1$. Then

$$f(z) = -\frac{z+1}{z-1} = \frac{\cos(\frac{\pi}{4})z + \sin(\frac{\pi}{4})}{-\sin(\frac{\pi}{4})z + \cos(\frac{\pi}{4})}$$

1c) An inversion in a circle $\mathcal{C}$ has the form $g(z) = m + \dfrac{r^2}{\overline{z-m}}$, where m is the center and r is the radius of $\mathcal{C}$. Again there are two cases:

(1): $g(\infty) = 1$ gives $m = 1$ and then $g(0) = -1$ gives $r^2 = 2$, i.e.

$$g(z) = 1 + \frac{2}{\overline{z}-1} = \frac{\overline{z}+1}{\overline{z}-1}.$$

(2): $g(\infty) = -1$ gives $m = -1$ and then $g(0) = 1$ again gives $r^2 = 2$, i.e.

$$g(z) = -1 + \frac{2}{\overline{z}+1} = \frac{-\overline{z}+1}{\overline{z}+1}.$$

Problem 2

2a) Substituting $\cos(\pi-\gamma) = -\cos\gamma$ and the formula for $\cos(\alpha+\beta)$ transforms the formula

$$(*)\qquad \cos(\alpha+\beta) - \cos(\pi-\gamma) = \sin\alpha\sin\beta(\cosh c - 1) \qquad \text{into}$$

$$\cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\gamma = \sin\alpha\sin\beta\cosh c - \sin\alpha\sin\beta.$$

This is clearly equivalent to the usual formulation of the second law of cosines.

In a triangle with angles α, β, γ the sum of two angles is less than π (= 2 right angles). Since cos is decreasing in $[0, \pi]$, it follows that the left hand side is positive if and only if $\alpha+\beta < \pi-\gamma$, or $\alpha+\beta+\gamma < \pi$. But $\cosh c > 1$

for $c > 0$, so this means that $\alpha + \beta + \gamma < \pi$ if and only if there is a $c > 0$ such that the stated second cosine relation holds. In terms of triangles; if there is a hyperbolic triangle with angles α, β, γ, then such a c exists (the side opposite γ), and hence $\alpha + \beta + \gamma < \pi$. Conversely, if $\alpha + \beta + \gamma < \pi$, we can solve this equation for c (and the other sides), and thus construct the triangle. Uniqueness follows, e. g. since the sides are uniquely determined.

Existence can also be proved by constructing the two rays $[0, 1)$ and $[0, \mathrm{e}^{i\alpha})$ in the Poincaré model, and then considering all rays $[r, q)$ where $r \in (0, 1)$, $\operatorname{Im} q > 0$ and $\angle(0\, r\, q) = \beta$. As r increases from 0 to 1, the latter ray will first intersect $[0, \mathrm{e}^{i\alpha})$ at an angle decreasing from $\pi - \alpha - \gamma$ to 0, and then not intersect at all. By continuity, at one point it intersects at the angle γ.

2b) By (a) there is a triangle T with angles $\pi/3, \pi/4$ and $\pi/4$, unique up to congruence. Joining six of these together at the $\pi/3$-vertex (the "center") in the obvious way constructs the hexagon. This is easiest to visualize if we let the center be the origin in the Poincaré disk model.

Using this model, one could also start by constructing the six rays from 0 to $\mathrm{e}^{2\pi ki/6}$, $k = 1, \ldots, 6$. Any circle with center 0 will cross these rays in vertices of regular hexagons, and when the Euclidean radius of this circle goes from 0 to 1, the vertex angle goes from $2\pi/3$ to 0. By continuity, one of the hexagons must have vertex angle $\pi/2$.

The area of each of the triangles is $\pi - (\pi/3 + \pi/4 + \pi/4) = \pi/6$, so the total area is π.

The side s of the hexagon is the side of T opposite to the angle $\pi/3$. It is determined by the second law of cosines:

$$\cos\frac{\pi}{3} = -\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4}\cosh s,$$

which gives $\cosh s = 2$, or $s = \ln(2 + \sqrt{2^2 - 1}) = \ln(2 + \sqrt{3})$.

The radius R of the circumscribed circle is the common length of the other sides of T, and is found for instance by using the hyperbolic sine law:

$$\frac{\sin R}{\sin\frac{\pi}{4}} = \frac{\sinh s}{\sin\frac{\pi}{3}}.$$

Using $\sinh s = \sqrt{\cosh^2 s - 1}$, this gives $\sinh R = \sqrt{2}$ and $R = \ln(\sqrt{2} + \sqrt{3})$.

Problem 3

3a) The standard way of parametrizing surfaces of revolution gives

$$r(u, v) = (u, \cosh u \cos v, \cosh u \sin v).$$

Then $r_u = (1, \sinh u \cos v, \sinh u \sin v)$, $\quad r_v = (0, -\cosh u \sin v, \cosh u \cos v)$. Hence $E = r_u \cdot r_u = 1 + \sinh^2 u = \cosh^2 u$, $\quad F = r_u \cdot r_v = 0$, and $G = r_v \cdot r_v = \cosh^2 u$.

The area of S_a is then

$$\iint_{S_a} \sqrt{EG - F^2}\, du\, dv = \int_0^{2\pi} [\int_0^a \cosh^2 u\, du] dv = \frac{\pi}{2}\sinh(2a) + \pi a.$$

3b) The Gauss map is a choice of unit normal vector N, considered as a map from S to the unit sphere S^2. For N we use the normalized vector product of basis vectors r_u and r_v and get:

$$N(r(u,v)) = \frac{r_u \times r_v}{|r_u \times r_v|} = \frac{(\sinh u, -\cos v, -\sin v)}{\cosh u}.$$

Assume that $N(r(u,v)) = N(r(u',v'))$. Comparing the first components, we see that then $\tanh u = \tanh u'$. But tanh is injective, so $u = u'$.

From the second and third components it now follows that $\cos v = \cos v'$ and $\sin v = \sin v'$, and then $v - v'$ must be a multiple of 2π. But then $r(u,v) = r(u',v')$.

To see what the image is, observe that $\tanh u$ can be anything in the interval $(-1,1)$, but not ± 1. However, if $x \neq \pm 1$ and $x^2 + y^2 + z^2 = 1$, we can set $x = \tanh u$ for some u. Then $y^2 + z^2 = 1 - \tanh^2 u = 1/\cosh u$, and we can find v such that $y = -\cos v/\cosh u, z = -\sin v/\cosh u$.

Thus the image of N is $S^2 - \{(\pm 1, 0, 0)\}$.

3c) To compute the Gaussian curvature we use the formula $K = \dfrac{eg - f^2}{EG - F^2}$.

First we need the second derivatives:

$r_{uu} = (0, \cosh u \cos v, \cosh u \sin v)$, $r_{uv} = (0, -\sinh u \sin v, \sinh u \cos v)$ and $r_{vv} = (0, -\cosh u \cos v, -\cosh v \sin v)$. Then

$$e = N \cdot r_{uu} = -1, \;\; f = N \cdot r_{uv} = 0 \;\text{ and }\; g = N \cdot r_{vv} = 1.$$

Thus $K(r(u,v)) = \dfrac{-1}{\cosh^4 u}$. The integral over S_a is

$$\iint_{S_a} K\, dA = \int_{u=0}^{a} \int_{v=0}^{2\pi} K(r(u,v))\sqrt{EG - F^2} du\, dv$$
$$= \int_{u=0}^{a} \int_{v=0}^{2\pi} \frac{-1}{\cosh^4 u} \cosh^2 u\, du\, dv = -2\pi \tanh a.$$

Clearly $\lim_{a\to\infty} \iint_{S_a} K\, dA = -2\pi$.

3d) Since the curve lies in a plane, both its tangent and acceleration vectors will lie in this plane. The curve will therefore be a geodesic if the surface normal also lies in the plane. But in the plane $x = 0$, the parameter $u = 0$, and then the x-component $\tanh u$ of N vanishes. Thus N indeed lies in the same plane.

Evidently all rotations about the x-axis (corresponding to the maps $(u,v) \mapsto (u, v + \theta)$ on parameters) are isometries of S. They also preserve the curves $\mathcal{C}_a$, hence also the geodesic curvature, since it is intrinsic. Therefore

$$\int_{\mathcal{C}_a} {k_g}^a\, ds = k_g^a \ell(\mathcal{C}_a) = 2\pi k_g \cosh a.$$

Substituting this into the Gauss-Bonnet theorem

$$\iint_{S_a} K\, dA + \int_{\mathcal{C}_a} {k_g}^a\, ds = 2\pi\chi(S_a)$$

gives $-2\pi \tanh a + 2\pi k_g^a \cosh a = 0$.

Hence $k_g^a = \dfrac{\tanh a}{\cosh a} = \dfrac{\sinh a}{\cosh^2 a}$.

Problem 4

Let R stand for either of the two components. Since the boundary curve is smooth and geodesic, the Gauss-Bonnet theorem takes its simplest form

$$\iint_R K\,dA = 2\pi\chi(R).$$

Since he curvature is negative, the left hand side is also negative. But if R is either a disk or a Möbius band, the right hand side is 0 or 2π. This is a contradiction.