MAT4510: Exam 2011, suggested solution

Problem 1

$f(z) = \frac{\sqrt{3}z-1}{z}$. Here $a = \sqrt{3}, b = -1, c = 1$ and $d = -0$. Moreover $(a+d)^2 = 3$, it follows that f is of elliptic type. $f(z) = z \Leftrightarrow z^2 - \sqrt{3}z + 1 = 0 \Leftrightarrow z = \frac{\sqrt{3}}{2} \pm \frac{i}{2}$. Let $h(z) = 2z - \sqrt{3}$. Then $h \in$Möb$^+(\mathbb{H})$ and $h(\frac{\sqrt{3}}{2} \pm \frac{i}{2}) = i$.

Consider $g = h \circ f \circ h^{-1}$. Now h, f and h^{-1} correspond to the matrices

$$\begin{bmatrix} 2 & -\sqrt{3} \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \sqrt{3} & -1 \\ 1 & -0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 2 \end{bmatrix}.$$

Put $g = h \circ f \circ h^{-1}$. So g corresponds to the matrix

$$\begin{bmatrix} 2 & -\sqrt{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & -0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$

So $g(z) = \frac{\sqrt{3}z-1}{z+\sqrt{3}} = \frac{\frac{-\sqrt{3}}{2}z+\frac{1}{2}}{\frac{-z}{2}-\frac{\sqrt{3}}{2}} = \frac{\cos\theta z+\sin\theta}{-\sin\theta z+\cos\theta}$ with $\theta = \frac{5\pi}{2}$, is an elliptic map on normal form conjugate to f.

Problem 2

The first hyperbolic law of cosines is:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos\alpha.$$

The hyperbolic law of sines is:

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}.$$

From the first and second hyperbolic law of cosines we get that $\cosh c = \cosh a \cosh b$ and thus $\cosh b = \frac{\cosh c}{\cosh a}$ and $\cos\beta = \sin\alpha \cosh b$ (since $\cos\gamma = 0$ ans $\sin\gamma = 1$). From the hyperbolic law of sines we get that $\sin\alpha = \frac{\sin\gamma}{\sinh c}\sinh a = \frac{\sinh a}{\sinh c}$, and we thus get that

$$\cos\beta = \sin\alpha \cosh b = \frac{\sinh a}{\sinh c}\frac{\cosh c}{\cosh a} = \frac{\sinh a}{\cosh a} \Big/ \frac{\sinh c}{\cosh c} = \frac{\tanh a}{\tanh c}.$$

Problem 3

The line $x = \frac{1}{2}\sqrt{2}$ and $|z| = 1$ will intersect at a point $\cos\theta + i\sin\theta$ where $\cos\theta = \frac{1}{2}\sqrt{2}$ so $\theta = \frac{\pi}{4}$. This means that the inner angle α at this vertex of R is $\frac{\pi}{2} - \frac{\pi}{4}$. The line $x = \frac{1}{2}\sqrt{2}$ and $|z| = \sqrt{2}$ will intersect at a point $\sqrt{2}(\cos\theta + i\sin\theta)$ where $\sqrt{2}\cos\theta = \frac{1}{2}\sqrt{2}$ so $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$. This means that the inner angle β at this vertex of R is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$. From symmetry it is clear that the two other inner angles of R is also α and β. Subdividing R into two hyperbolic triangles and using the hyperbolic area formulae for such triangles, we see that the area of R is $2\pi - (\alpha+\alpha+\beta+\beta) = 2\pi - \frac{11}{6}\pi = \frac{\pi}{6}$. (This area may also be found by integration.)

Problem 4

a)

$$\mathbf{x}_u = (-v\sin u, v\cos u, u), \mathbf{x}_v = (\cos u, \sin u, 0), \mathbf{x}_u \times \mathbf{x}_v = (-u\sin u, u\cos u, -v).$$

So $E = \mathbf{x}_u \cdot \mathbf{x}_u = u^2 + v^2$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v = 1$. The first fundamental form is thus given by

$$ds^2 = (u^2 + v^2)du^2 + dv^2.$$

Moreover $N = \frac{(u \sin u, -u \cos u, v)}{\sqrt{u^2+v^2}}$, and we get that

$$e = \mathbf{x}_{uu} \cdot N = (-v \cos u, -v \sin u, 1) \cdot N = \frac{v}{\sqrt{u^2 + v^2}},$$

$$f = \mathbf{x}_{uv} \cdot N = (-\sin u, \cos u, 0) \cdot N = \frac{-u}{\sqrt{u^2 + v^2}} \text{ and } g = \mathbf{x}_{vv} \cdot N = \mathbf{0} \cdot N = 0.$$

It follows that the curvature is given by

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-u^2}{(u^2 + v^2)^2}.$$

b) Let E, F and G be the coefficients of the first fundamental form of S' with respect to the parametrization of $\mathbf{y}$. Then

$$E = \mathbf{y}_u \cdot \mathbf{y}_u = (-v \sin u, v \cos u, 0) \cdot (-v \sin u, v \cos u, 0) = v^2,$$

$$F = \mathbf{y}_u \cdot \mathbf{y}_v = (-v \sin u, v \cos u, 0) \cdot (\cos u, \sin u, 1) = 0,$$

$$G = \mathbf{y}_v \cdot \mathbf{y}_v = (\cos u, \sin u, 1) \cdot (\cos u, \sin u, 1) = 2.$$

Let E', F' and G' be the coefficients of the first fundamental form of U with respect to the parametrization

$$\mathbf{z}(u, v) = (\sqrt{2} v \cos \frac{u}{\sqrt{2}}, \sqrt{2} v \sin \frac{u}{\sqrt{2}}).$$

Then

$$E' = \mathbf{z}_u \cdot \mathbf{z}_u = (-v \sin \frac{u}{\sqrt{2}}, v \cos \frac{u}{\sqrt{2}}) \cdot (-v \sin \frac{u}{\sqrt{2}}, v \cos \frac{u}{\sqrt{2}}) = v^2,$$

$$F = \mathbf{z}_u \cdot \mathbf{z}_v = (-v \sin \frac{u}{\sqrt{2}}, v \cos \frac{u}{\sqrt{2}}) \cdot (\sqrt{2} \cos \frac{u}{\sqrt{2}}, \sqrt{2} \sin \frac{u}{\sqrt{2}}) = 0,$$

$$G' = \mathbf{z}_v \cdot \mathbf{z}_v = (\sqrt{2} \cos \frac{u}{\sqrt{2}}, \sqrt{2} \sin \frac{u}{\sqrt{2}}) \cdot (\sqrt{2} \cos \frac{u}{\sqrt{2}}, \sqrt{2} \sin \frac{u}{\sqrt{2}}) = 2.$$

since $E = E', F = F'$ and $G = G'$ and f also is a diffeomorphism onto U, it follows that F is an isometry. Since $\mathbf{R}^2$ and thus U have Gaussian curvature equal 0 and f is an isometry, it follows from theorem Egregium that S' must have constant curvature equal 0. Since the curvature of S $\frac{-u^2}{(u^2+v^2)^2}$ is never 0, it follows again from theorem Egregium that there cannot exist any local isometry between S and S'.

c) The curves $u = constant$ are straight lines on S' and therefore geodesics. If $av(t) \cos \frac{u(t)}{\sqrt{2}} + bv(t) \sin \frac{u(t)}{\sqrt{2}} = c$, then $a\sqrt{2}v(t) \cos \frac{u(t)}{\sqrt{2}} + b\sqrt{2}v(t) \sin \frac{u(t)}{\sqrt{2}} = c\sqrt{2}$. This means that $\mathbf{z}(u, v)$ maps the curves $(u(t), v(t)$ to the intersection of U and the straight lines in $\mathbb{R}^2$ with equation $ax + by = \sqrt{2}c$. Since straight lines are geodesics, the curves $(u(t), v(t))$ correspond to geodesics in U. Since isometries map geodesics to geodesic and f and thus f^{-1} are isometries, the curves also correspond to geodesics on S'.

Problem 5

a) Since the line α has equation $y = x + 1$ and the circle $|z + 1| = 2$ has equation $(x+1)^2 + y^2 = 4$, the circle and the line intersect when $2(x+1)^2 = 4, x = \sqrt{2} - 1$ and $y = \sqrt{2}$. The region R is therefore given by $x + 1 \leq y \leq \sqrt{4 - (x+1)^2}$, $0 \leq x \leq \sqrt{2} - 1$. So the hyperbolic area of R is given by

$$A(R) = \int\int_R \frac{dxdy}{y^2} = \int_0^{\sqrt{2}-1} \int_{x+1}^{\sqrt{4-(x+1)^2}} \frac{dydx}{y^2} = \int_0^{\sqrt{2}-1} \left[-\frac{1}{y} \right]_{x+1}^{\sqrt{4-(x+1)^2}} =$$

$$\int_0^{\sqrt{2}-1} (\frac{1}{x+1} - \frac{1}{\sqrt{4-(x+1)^2}})dx = \left[\ln(x+1) - \arcsin \frac{x+1}{2} \right]_0^{\sqrt{2}-1}$$

$$= \frac{1}{2} \ln 2 - \arcsin \frac{\sqrt{2}}{2} + \arcsin \frac{1}{2} = \frac{1}{2} \ln 2 - \frac{\pi}{4} + \frac{\pi}{6} = \frac{1}{2} \ln 2 - \frac{\pi}{12}.$$

b) The hyperbolic arc-length of the curve is given by

$$s(t_0) = \int_0^{t_0} \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt = \int_0^{t_0} \frac{\sqrt{2}}{t+1} dt = \left[\sqrt{2} \ln(t+1) \right]_0^{t_0} = \sqrt{2} \ln(t_0 + 1).$$

We have $s = s(t) = \sqrt{2} \ln(t+1) \Rightarrow t = e^{\frac{s}{\sqrt{2}}} - 1$. So $\alpha(s) = (e^{\frac{s}{\sqrt{2}}} - 1) + ie^{\frac{s}{\sqrt{2}}}$ is the parametrization of α by arc-length.

c) With $x(s) = e^{\frac{s}{\sqrt{2}}} - 1$ and $y(s) = e^{\frac{s}{\sqrt{2}}}$ we get that

$$D\alpha''(s) = (x'' - \frac{2}{y} x'y') + i(y'' + \frac{1}{y}(x')^2 - \frac{1}{y}(y')^2) = \frac{1}{2} e^{\frac{s}{\sqrt{2}}}(-1 + i).$$

Since $D\alpha''(s)$ and $n_\alpha(s)$ point in the same direction, we get that $k_g(s) = ||D\alpha''(s)||_{\alpha(s)} = \frac{1}{2} \frac{\sqrt{2} e^{\frac{s}{\sqrt{2}}}}{e^{\frac{s}{\sqrt{2}}}} = \frac{1}{\sqrt{2}}$. So

$$\int_{\alpha_1} k_g ds = \frac{1}{\sqrt{2}} l(\alpha_1) = \frac{1}{\sqrt{2}} \sqrt{2} \ln((\sqrt{2} - 1) + 1) = \frac{1}{2} \ln 2.$$

d) The non-smooth points of R is $p_1 = (\sqrt{2} - 1) + i\sqrt{2}$, $p_2 = \sqrt{3}i$ and $p_3 = i$. Let ϵ_1, ϵ_2 and ϵ_3 be the outer angles at p_1, p_2 and p_3 respectively. We see immeadiately that $\epsilon_1 = \frac{\pi}{2}$ and $\epsilon_3 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. The unit normal of $|z + 1| = 2$ at p_2 is $\frac{1}{2}(1, \sqrt{3})$. This normal vector makes and angle $\frac{\pi}{6}$ with the imaginary axis (since $\cos \frac{\pi}{6} = \frac{1}{2}\sqrt{3}$). The outer angle ϵ_2 is therefore equal to $\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$. Since the boundary of R is the union of α_1 and two other curves which are H-lines and therefore geodesics, we get that $\int_{\partial R} k_g(s)ds = \int_{\alpha_1} k_g(s)ds$. Recall that the curvature in $\mathbb{H}$ is-1. From a) and c) and above we get that

$$\int\int_R K dA + \int_{\partial R} k_g(s)ds + \epsilon_1 + \epsilon_2 + \epsilon_3 = -A(R) + \int_{\partial R} k_g(s)ds + \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$= -(\frac{1}{2} \ln 2 - \frac{\pi}{12}) + \frac{1}{2} \ln 2 + \frac{\pi}{2} + \frac{2\pi}{3} + \frac{3\pi}{4} = 2\pi.$$

On the other hand, it is clear that R is homeomorphic to a triangle and therefore $\chi(R) = 1 - 3 + 3 = 1$, so $2\pi\chi(R) = 2\pi$. This verifies the Gauss-Bonnet Theorem for this region R in $\mathbb{H}$.