MAT4510: Exam 2013, suggested solution

Problem 1

a) $f(z) = \frac{2}{2-z}$. $f(z) = z \Leftrightarrow z^2 - 2z + 2 = 0 \Leftrightarrow z = 1 \pm i$. Since f has complex fixpoints, it follows that f is elliptic.

Let $p = 1 + i$ and $h(z) = \frac{z - \mathrm{Re}p}{\mathrm{Im}p} = z - 1$. Then $h \in$Möb$^+(\mathbb{H})$ and $h(p) = i$.

Consider $g = h \circ f \circ h^{-1}$. Now h, f and h^{-1} correspond to the matrices

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Put $g = h \circ f \circ h^{-1}$. So g corresponds to the matrix

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

So $g(z) = \frac{z+1}{-z+1} = \frac{\frac{\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}}{\frac{-\sqrt{2}}{2}z + \frac{\sqrt{2}}{2}} = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$ with $\theta = \frac{\pi}{4}$, is an elliptic map on normal form conjugate to f.

b) $f(z) = \frac{4\bar{z}+20}{5\bar{z}+4}$. Let $z = x + iy$. $f(z) = z \Leftrightarrow 5|z|^2 + 4(z - \bar{z}) - 20 = 0 \Leftrightarrow 5(x^2 + y^2) = 20$, $8y = 0 \Leftrightarrow z = \pm 2$. So the fixpoints are $z = \pm 2$. Let $h(z) = \frac{z-2}{z+2}$. Then $h \in$Möb$^+(\mathbb{H})$ and $h(2) = 0$, $h(-2) = \infty$. Let $f' = h \circ f \circ h^{-1}$. Then f' corresponds to the matrix:

$$\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 20 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -24 & 0 \\ 0 & 56 \end{bmatrix}.$$

So $f'(z) = -\frac{24}{56}\bar{z} = -\frac{3}{7}\bar{z}$. Put $g'(z) = -\bar{z}$ and $k'(z) = \frac{3}{7}z$. Then g' is an inversion, $k' \in$Möb$^+(\mathbb{H})$ and $g' \circ k' = k' \circ g' = f'$. Put $g = h^{-1} \circ g' \circ h$ and $k = h^{-1} \circ k' \circ h$. It follows that g and k will have the desired properties.

Now g corresponds to the matrix

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}.$$

So $g(z) = \frac{8}{2\bar{z}} = \frac{4}{\bar{z}}$. We see that when $|z| = 2$ then $g(z) = \frac{4z}{\bar{z}z} = \frac{4z}{|z|^2} = \frac{4z}{4} = z$. So g is the inversion in the $\mathbb{H}$-line $|z| = 2$.

k corresponds to the matrix

$$\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 16 \\ 4 & 20 \end{bmatrix}.$$

So $k(z) = \frac{20z+16}{4z+20} = \frac{5z+4}{z+5}$.

Problem 2

a)

$$A = \int_0^1 \int_1^{-x+2} \frac{dy\,dx}{y^2} = \int_0^1 \left[-\frac{1}{y}\right]_1^{-x+2} dx = \int_0^1 (1 - \frac{1}{2-x})dx = [x + \ln(2-x)]_0^1 = 1 - \ln 2.$$

b) From the first hyperbolic law of cosine, we have $\cosh b = \cosh a \cosh c - \sinh a \sinh c \cos\beta$ hence

$$\cos\beta = \frac{\cosh a \cosh c - \cosh b}{\sinh a \sinh c}.$$

Similarly, we get that

$$\cos\gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}.$$

Using that $b = c$, we get that

$$\cos\gamma = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b} = \frac{\cosh a \cosh c - \cosh b}{\sinh a \sinh c} = \cos\beta.$$

Since $\beta, \gamma \in (0, \pi)$ and cos is 1-1 in this interval, we must have $\beta = \gamma$.

c) Let us denote the angles of T at i, z_2 and z_1 by α, β and γ respectively . Let the hyperbolic length of the opposite sides be a, b and c. Then $\cosh b = \frac{1}{2}(\sqrt{2} + \sqrt{3} + \frac{1}{\sqrt{2}+\sqrt{3}}) = \sqrt{3} = \cosh c$. Furthermore $\sinh b = \sqrt{\cosh^2 b - 1} = \sqrt{2} = \sinh c$. It is clear that the angle α is the same as the angle at i of the Euclidean triangle with vertices $-1, i, 1$. This angle is equal $\frac{\pi}{2}$ (since the two other angles both are $\frac{\pi}{4}$). Now the first hyperbolic law of cosine give us (since $\alpha = \frac{\pi}{2}$ hence $\cos\alpha = 0$)

$$\cosh(z_1, z_2) = \cosh a = \cosh b \cosh c = 3.$$

d) From c) we get that $\sinh a = \sqrt{9-1} = 2\sqrt{2}$. From b) and c) we get that

$$\cos\beta = \frac{\cosh a \cosh c - \cosh b}{\sinh a \sinh c} = \frac{3\sqrt{3} - \sqrt{3}}{4} = \frac{\sqrt{3}}{2}.$$

It follows that $\beta = \frac{\pi}{6}$ and from b) we also get that $\gamma = \frac{\pi}{6}$. From the area formulae of a geodesic triangle, we get that the area of T is equal to $\pi - \frac{\pi}{2} - 2\frac{\pi}{6} = \frac{\pi}{6}$.

Problem 3

a)

$$\mathbf{x}_u = (1, 0, v), \mathbf{x}_v = (0, 1, u)).$$

So $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1 + v^2$, $F = \mathbf{x}_u \cdot \mathbf{x}_v = uv$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v = 1 + u^2$. The first fundamental form is thus given by

$$ds^2 = (1 + v^2)du^2 + 2uvdudv + (1 + u^2)dv^2.$$

Moreover

$$\mathbf{y}_u = (-\cosh v \sin u, \cosh v \cos u, 0), \mathbf{y}_v = (\sinh v \cos u, \sinh v \sin u, \cosh v).$$

So $E = \mathbf{y}_u \cdot \mathbf{y}_u = \cosh^2 v$, $F = \mathbf{y}_u \cdot \mathbf{y}_v = 0$ and $G = \mathbf{y}_v \cdot \mathbf{y}_v = \cosh^2 v + \sinh^2 v$. The first fundamental form is thus given by

$$ds^2 = \cosh^2 v du^2 + (\cosh^2 v + \sinh^2 v)dv^2.$$

b) We have
$\mathbf{x}_u \times \mathbf{x}_v = (-v, -u, 1)$, $N_S = \frac{(-v,-u,1)}{\sqrt{1+u^2+v^2}}$.
Moreover $x_{uu} = \mathbf{0}$, $x_{uv} = (0, 0, 1)$ and $x_{vv} = \mathbf{0}$. Hence $e = g = 0$ and $f = \mathbf{x}_{uv} \cdot N_S = \frac{1}{\sqrt{1+u^2+v^2}}$. It follows that

$$K_S = \frac{eg - f^2}{EG - F^2} = \frac{-1}{(1 + u^2 + v^2)^2}.$$

Similarly
$\mathbf{y}_u \times \mathbf{y}_v = (\cosh^2 v \cos u, \cosh^2 v \sin u, -\cosh v \sinh v)$ so $N_{S'} = \frac{(\cosh v \cos u, \cosh v \sin u, -\sinh v)}{\sqrt{\cosh^2 v + \sinh^2 v}}$.

Furthermore $\mathbf{y}_{uu} = (-\cosh v \cos u, -\cosh v \sin u, 0)$, $\mathbf{y}_{uv} = (-\sinh v \sin u, \sinh v \cos u, 0)$ and $\mathbf{y}_{vv} = (\cosh v \cos u, \cosh v \sin u, \sinh v)$.

$$e = \mathbf{y}_{uu} \cdot N_{S'} = \frac{-\cosh^2 v}{\sqrt{\cosh^2 v + \sinh^2 v}},$$
$$f = \mathbf{x}_{uv} \cdot N_{S'} = 0 \text{ and } \; g = \mathbf{x}_{vv} \cdot N = \frac{\cosh^2 v - \sinh^2 v}{\sqrt{\cosh^2 v + \sinh^2 v}} = \frac{1}{\sqrt{\cosh^2 v + \sinh^2 v}}$$

Now, for S', we have that $EG - F^2 = \cosh^2 v(\cosh^2 v + \sinh^2 v)$. It follows that the curvature is given by

$$K_{S'} = \frac{eg - f^2}{EG - F^2} = \frac{-1}{(\cosh^2 v + \sinh^2 v)^2}.$$

Since $K_S = \frac{-1}{(1+u^2+v^2)^2}$, it follows that $(0,0,0)$ is the only point in S with Gaussian curvature equal -1. Now if there exists some isometry $\phi : U \to \phi(U) \subset S'$, it follows from Theorem Egregium that $\phi(0,0,0)$ must have Gaussian curvature equal -1. Now the points in S' with Gaussian curvature equal -1 are the points on the circle given by $v = 0$. So $\phi(U)$ must contain more than one point (actually infinitely many points) with Gaussian curvature equal -1. Since $(0,0,0)$ is the only point in S with Gaussian curvature equal -1 and ϕ preserves the Gaussian curvature (again by Theorem Egregium), we will get a contradiction. It follows that we cannot have a local isometry defined around $(0,0,0)$.

c) We have that $\alpha'(t) = (1,1,2t)$, $\alpha''(t) = (0,0,2)$ and $N_S(\alpha(t)) = \frac{(-t,-t,1)}{\sqrt{1+2t^2}}$. So

$$\det \begin{bmatrix} \alpha'(t) \\ \alpha''(t) \\ N_S(\alpha(t)) \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 2t \\ 0 & 0 & 2 \\ \frac{-t}{\sqrt{1+2t^2}} & \frac{-t}{\sqrt{1+2t^2}} & \frac{1}{\sqrt{1+2t^2}} \end{bmatrix} = 0,$$

and it follows that $\alpha(t)$ is a geodesic. Furthermore, $\beta(t)$ and $\gamma(t)$ are straight lines and it follows that these curves also are geodesics.

d) The boundary of R_a consists of the curves $\beta(t) = (t,0,0), t \in [0,a]$, $\gamma(t) = (a,t,at), t \in [0,a]$ and $\alpha(t) = (a-t, a-t, (a-t)^2), t \in [0,a]$. It follows from c) that these curves all are geodesics.
Now $\cos\eta_1 = \frac{\beta'(0)\cdot(-\alpha'(a))}{||\beta'(0)||\,||\alpha'(0)||} = \frac{(1,0,0)\cdot(1,1,0)}{\sqrt{2}} = \frac{1}{\sqrt{2}}$, and $\cos\eta_2 = \frac{-\beta'(a)\cdot(\gamma'(0))}{||\beta'(a)||\,||\gamma'(0)||} = \frac{(-1,0,0)\cdot(0,1,a)}{\sqrt{1+a^2}} = 0$. It is geometrically obvious that η_1 and η_2 are angles in the interval $[0,\pi]$ (where cos is 1-1), hence $\eta_1 = \frac{\pi}{4}$ and $\eta_2 = \frac{\pi}{2}$. Finally $\cos\eta_3 = \frac{\alpha'(0)\cdot(-\gamma'(a))}{||\alpha'(0)||\,||\gamma'(a)||} = \frac{(-1,-1,-2a)\cdot(0,-1,-a)}{\sqrt{2+4a^2}\sqrt{1+a^2}} = \frac{1+2a^2}{\sqrt{2+4a^2}\sqrt{1+a^2}} = \sqrt{\frac{1+2a^2}{2+2a^2}}$. Let ϵ_i $i = 1,2,3$ be the exteriour angles at $(0,0,0)$, $(a,0,0)$ and (a,a,a^2). It is clear that $\epsilon_1 = \frac{3\pi}{4}$ and that $\epsilon_2 = \frac{\pi}{2}$. By a) and b) we have that

$$\int\!\!\int_{R_a} K_S dA = \int\!\!\int_{T_a} K_S(u,v)\sqrt{EG - F^2} du dv = \int\!\!\int_{T_a} \frac{-du dv}{(1+u^2+v^2)^{\frac{3}{2}}},$$

moreover R_a is homemorphic to the triangle T_a hence $\chi(R_a) = 2\pi$. So by the Gauss Bonnet Theorem we get that (using that the boundary of T_a consists of geodesics)

$$\int\int_{T_a} \frac{dudv}{(1+u^2+v^2)^{\frac{3}{2}}} = \epsilon_1 + \epsilon_2 + \epsilon_3 - 2\pi.$$

Since $\lim_{a\to\infty} \cos\eta_3 = \lim_{a\to\infty} \sqrt{\frac{1+2a^2}{2+2a^2}} = 1$, $\lim_{a\to\infty} \eta_3 = 0$ and $\lim_{a\to\infty} \epsilon_3 = \pi$. We thus get that

$$\lim_{a\to\infty} \int\int_{T_a} \frac{dudv}{(1+u^2+v^2)^{\frac{3}{2}}} = \frac{3\pi}{4} + \frac{\pi}{2} + \pi - 2\pi = \frac{\pi}{4}.$$