MAT4510: Sugested solutions Fall 2010

Problem 1

Here, $a = d = \sqrt{2}$ and $b = c = 1$. So $ad - bc = 1$ and $(a+d)^2 = 8 > 4$, so f is of hyperbolic type.

$$f(z) = z \Leftrightarrow \sqrt{2}z + 1 = z^2 + \sqrt{2}z \Leftrightarrow z^2 = 1 \Leftrightarrow z = \pm 1.$$

So f has fixpoints ± 1. Let $g(z) = \frac{z+1}{1-z}$. Then $g \in \text{Möb}^+(\mathbb{H})$ and $g(-1) = 0$, $g(1) = \infty$. Matrices associated to f, g and g^{-1} are

$$\begin{bmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

respectively. So a matrix associated to $h = g \circ f \circ g^{-1}$ is consequently

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2(\sqrt{2}+1) & 0 \\ 0 & 2(\sqrt{2}-1) \end{bmatrix}.$$

f is consequently conjugate to the map $h(z) = g \circ f \circ g^{-1}(z) = \frac{2(\sqrt{2}+1)z}{2(\sqrt{2}-1)} = (3+2\sqrt{2})z$, which a hyperbolic map of normal form.

Problem 2

a) Since $E = G = \frac{1}{y^2}$ and $F = 0$, the equations for Γ_{ij}^k becomes (with $u = x$ and $v = y$)

$$\begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} y^2 & 0 \\ 0 & y^2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{y^3} & 0 \\ \frac{1}{y^3} & 0 & -\frac{1}{y^3} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{y} & 0 \\ \frac{1}{y} & 0 & -\frac{1}{y} \end{bmatrix}.$$

This gives $\Gamma_{11}^1 = \Gamma_{22}^1 = \Gamma_{12}^2 = 0$, and $\Gamma_{12}^1 = -\Gamma_{11}^2 = \Gamma_{22}^2 = -\frac{1}{y}$. We thus get that

$$D\beta''(s) = (x'' - 2\frac{x'y'}{y}) + (y'' + \frac{(x')^2}{y} - \frac{(y')^2}{y})i.$$

(Here $T_z\mathbb{H}$ is identified with $\mathbb{C}$ and $\mathbf{x}_x$ and $\mathbf{x}_y$ are identified with 1 and i respectively.) parameterization b) For any two points on C_{y_0} there is a Möbius transformation of

type $f(z) = z + a$, $a \in \mathbb{R}$ mapping one to the other. Since such transformations are isometries preserving the normal orientation, and the geodesic curvature is preserved under such isometries, k_g must be constant along C_{y_0}.

Let $z(t) = t + y_0 i$, $s(t) = \int_0^t ||z'(\tau)||d\tau = \int_0^t \frac{d\tau}{y_0} = \frac{t}{y_0}$. So $t = y_0 s$ and $\beta(s) = y_0 s + y_0 i$ is a parameterization of C_{y_0} by (hyperbolic) arc-length. From the solution of a) we see that $D''\beta(s) = y_0 i$. Since $y_0 i$ is a the unit-normal vector along C_{y_0}, we get that $k_g = 1$ along C_{y_0}.

c) $\partial R = \alpha_1 \cup \alpha_2$ where α_1 is contained in the circle $\{|z| = 1\}$, hence α_1 is a geodesic in $\mathbb{H}$, and α_2 is contained in $C_{\frac{\sqrt{2}}{2}}$. Since $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, it easy to see that the inner angles η_i, $i = 1, 2$ at the vertices $\pm\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ are equal $\frac{\pi}{4}$ hence the outer angles ϵ_i are both equal $\frac{3\pi}{4}$. The arc-length of α_2 is by the calculation in b) equal

$l = \frac{\sqrt{2}}{\frac{\sqrt{2}}{2}} = 2$, and we consequently get that $\int_{\alpha_2} k_g ds = k_g l = 2$. Now Gauss-Bonnet Theorem give us

$$\int\int_R K dA + \int_{\partial R} k_g ds + \epsilon_1 + \epsilon_2 = -A(R) + 2 + \frac{3\pi}{2} = 2\pi\chi(R) = 2\pi.$$

We thus get that $A(R) = 2 - \frac{\pi}{2}$. (Here we use that R obviously is homemorphic to a disc, hence $\chi(R) = 1$.)

d) Using that $dA = \frac{dxdy}{y^2}$, we get that

$$A(R) = \int\int_R dA = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{\frac{\sqrt{2}}{2}}^{\sqrt{1-x^2}} \frac{dydx}{y^2} = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \sqrt{2}dx - \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1-x^2}} dx$$
$$= 2 - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta = 2 - \frac{\pi}{2}.$$

Problem 3

Let α, β, γ be the angles at the vertices $ri, -r, r$ respectively. From the symmetry properties of T, it is easy to see that $\frac{\alpha}{2} = \beta = \gamma$. So we must have that $4\beta = \frac{2\pi}{3}$, and we get that $\alpha = \frac{\pi}{3}$ and $\beta = \gamma = \frac{\pi}{6}$. From the second law of cosine, we thus get that

$$\frac{1}{2} = -(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \cosh(d_{\mathbb{D}}(-r, r)) \Rightarrow \cosh(d_{\mathbb{D}}(-r, r)) = 5,$$

and we get that $\cosh a = \cosh(d_{\mathbb{D}}(-r, r)) = 1 + \frac{2|r-(-r)|^2}{(1-r^2)^2} = 1 + \frac{8r^2}{(1-r^2)^2} = 5$, $r^4 - 4r^2 + 1 = 0$ which implies that $r^2 = 2 \pm \sqrt{3}$. Here we must have $r^2 < 1$, and we get that $r = \sqrt{2 - \sqrt{3}}$.

Problem 4

a) Let the given parameterization be $\alpha(u, v)$ then

$$\alpha_u = (-a \sin u \cos v, -a \sin u \sin v, b \cos u),\ \ \alpha_v = (-a \cos u \sin v, a \cos u \cos v, 0),$$

and we get that $E = a^2 \sin^2 u + b^2 \cos^2 u, F = 0$ and $G = a^2 \cos^2 u$.

b) We get that $\alpha_u \times \alpha_v = (-ab \cos^2 u \cos v, -ab \cos^2 u \sin v, -a^2 \cos u \sin u)$, and we get that the unit surface-normal is equal

$$N(u, v) = \frac{(-b \cos u \cos v, -b \cos u \sin v, -a \sin u)}{\sqrt{b^2 \cos^2 u + a^2 \sin^2 u}}.$$

Moreover we get that

$$e = \alpha_{uu} \cdot N = ((-a \cos u \cos v, -a \cos u \sin v, -b \sin u) \cdot N = \frac{ab}{\sqrt{b^2 \cos^2 u + a^2 \sin^2 u}}$$

$$f = \alpha_{uv} \cdot N = (a \sin u \sin v, -a \sin u \cos v, 0) \cdot N = 0$$

$$g = \alpha_{vv} \cdot N = (-a \cos u \cos v, -a \cos u \sin v, 0) \cdot N = \frac{ab \cos^2 u}{\sqrt{b^2 \cos^2 u + a^2 \sin^2 u}}.$$

The curvature of the surface is given by

$$K = \frac{eg - f^2}{EG - F^2} = \frac{b^2}{(b^2 \cos^2 u + a^2 \sin^2 u)^2}.$$

c) The surface is a regular surface of rotation, obtained by rotating the ellipse $\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$ around the $z-axis$. Such a surface is obviously homeomorphic to a sphere, and S has consequently Euler characteristic equal 2. Letting $(u,v)\in[0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]$ we get a parameterization of the whole of S , and this parameterization is one-to-one on the interior of $\in[0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]$. The Gauss-Bonnet Theorem implies that

$$\int\int_S KdA=\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K\sqrt{EG-F^2}dudv=$$

$$2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{ab^2\cos u\,du}{(b^2\cos^2 u+a^2\sin^2 u)^{\frac{3}{2}}}=2\pi\chi(S)=4\pi,$$

and consequently that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{ab^2\cos u\,du}{(b^2\cos^2 u+a^2\sin^2 u)^{\frac{3}{2}}}=2.$$

d) In general, when $\alpha(t)$ is a parametrized curve on a regular surface, α is a geodesic if and only if $\alpha''(t)$ is a vector in the plane spanned by $\alpha'(t)$ and $N(\alpha(t))$ for each t (where $N(\alpha(t))$ is the surface normal along α). When the curve is a plane curve and the curve is not a line, α' and α'' are always linearly independent and will therefore (for each t) span this plane (or more precise, span the plane through the origin we get by a suitable translation), and α is consequently a geodesic if and only if $N(\alpha(t))$ is a vector in this plane for each t. The curves given by $v=constant$ is the intersection of the plane, $y=(\tan v)x$ and the the surface. Calculating, we get that $\mathbf{x}_u\times\mathbf{x}_v=(-g(u)h'(u)\cos v,-g(u)h'(u)\sin v,g(u)g'(u))$. Since this vector is parallel to N, and we se that for all u this vector is a vector in the plane $y=(\tan v)x$, the curve is consequently a geodesic. When $u=constant$ the curve is the intersection of the plane $z=h(u)$ and the surface. Then $\mathbf{x}_u\times\mathbf{x}_v$ is a vector in this plane and the curve is a geodesic, if and only if $g(u)g'(u)=0$, if and only if $g'(u)=0$ (since $g(u)>0$).