Problem 1

1a.

$$m(i) = \frac{i-0}{i-1} \cdot \frac{-1-1}{-1-0} = \frac{2i}{i-1} = 1 - i\,.$$

Since $i \in \mathbb{H}$ and $m(i) \notin \mathbb{H}$ we cannot have $m(\mathbb{H}) = \mathbb{H}$.

1b. The map m takes the $\bar{\mathbb{C}}$-circle through x_1, x_2 and x_3 onto the $\bar{\mathbb{C}}$-circle through 1, 0 and ∞, i.e., maps $\bar{\mathbb{R}}$ onto $\bar{\mathbb{R}}$. Hence it maps $\bar{\mathbb{C}} \setminus \bar{\mathbb{R}} = \mathbb{H} \cup (-\mathbb{H})$ onto itself, where $-\mathbb{H}$ denotes the lower half-plane. By continuity, $m(\mathbb{H})$ is either $\mathbb{H}$ or $-\mathbb{H}$. Now

$$m(z) = \frac{z-x_2}{z-x_3} \cdot \frac{x_1-x_3}{x_1-x_2} = \frac{az+b}{cz+d}$$

with $a = x_1 - x_3$, $b = -x_2(x_1-x_3)$, $c = x_1 - x_2$ and $d = -x_3(x_1-x_2)$. Here

$$m(i) = \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{(ci+d)(-ci+d)} = \frac{(ac+bd)+i(ad-bc)}{c^2+d^2}$$

lies in $\mathbb{H}$ if and only if $ad - bc > 0$, i.e., if and only if

$$(x_1-x_3)(-x_3)(x_1-x_2) - (-x_2)(x_1-x_3)(x_1-x_2) = (x_1-x_2)(x_1-x_3)(x_2-x_3) > 0\,.$$

The condition is that the product $(x_1-x_2)(x_1-x_3)(x_2-x_3)$ is positive.

Problem 2

2a. The points $A = w$ and $B = w^2$ have $|A| = 1$ and $|B| = 1$, hence lie on the Euclidean circle of radius 1 with center 0. The points $A = w$ and $C = 0$ have $|A-1| = 1$ and $|C-1| = 1$, hence lie on the Euclidean circle of radius 1 with center $+1$. The points $B = w^2$ and $C = 0$ have $|B+1| = 1$ and $|C+1| = 1$, hence lie on the Euclidean circle of radius 1 with center -1.

The angle $\alpha = \angle BAC$ is the angle at w between the Euclidean unit circles centered at 0 and $+1$. It equals the angle $\angle 0w1$ between the radii meeting at w of these two circles, hence equals $\pi/3$. By symmetry about the imaginary axis, the angle $\beta = \angle ABC$ is also $\pi/3$. The angle $\gamma = \angle ACB$ at the ideal vertex is 0. Hence the area of $\triangle ABC$ is

$$\pi - (\alpha + \beta + \gamma) = \pi - (\pi/3 + \pi/3 + 0) = \pi/3\,.$$

2b. The fixed points in $\bar{\mathbb{C}}$ of m_1 are the z with $-1/z = z$, i.e., with $z^2 = -1$, so $z = \pm i$. Hence m_1 has exactly one fixed point in $\mathbb{H}$, so m_1 is of elliptic type. We have $m_1(A) = -1/w = w^2 = B$ and $m_1(D) = -1/i = i = D$, so the image of $[A, D]$ is $[B, D]$. We have $m_1(B) = A$ and $m_1(C) = \infty$, so the image of $\triangle ABC$ is the ideal triangle $\triangle BA\infty$.

2c. The fixed points in $\bar{\mathbb{C}}$ of m_2 are the z with $z/(z+1) = z$, i.e., with $z + 1 = 1$, so $z = 0$. The only fixed point lies in $\bar{\mathbb{R}}$, so m_2 is of parabolic type. We have $m_2(B) = w^2/(w^2+1) = w = A$ and $m_2(C) = 0 = C$, so the image of $\overrightarrow{BC}$ is the ray $\overrightarrow{AC}$. We have

$$m_2(A) = w/(w+1) = (3 + i\sqrt{3})/6 = E\,,$$

so the image of $\triangle ABC$ is the ideal triangle $\triangle EAC$.

2d. The edges $a = [A, D]$, $[D, B]$, $b = [B, C]$ and $[C, A]$ of the quadrangle F are identified according to the pattern $W = aa^{-1}bb^{-1}$, so $M = F/{\sim} \cong D^2/aa^{-1}bb^{-1} \cong D^2/aa^{-1} \# D^2/bb^{-1} \cong S^2 \# S^2 \cong S^2$. Hence $M \cong M_0$ is the orientable surface S^2 of genus $g = 0$.

2e. No, M does not admit a hyperbolic structure. In the Gauss-Bonnet formula

$$\iint_M K\,dA = 2\pi\chi(M)$$

the Gaussian curvature of a hyperbolic closed surface is -1 at all points, so the Euler characteristic must be negative. This is not the case for $M \cong M_0 = S^2$, which has Euler characteristic $\chi(S^2) = 2$.

Yes, M' does admit a hyperbolic structure. Let $F' = F \setminus \{A, B, C, D\}$ be the complement of the four vertices in F, and consider the union

$$P = F' \cup m_1(F') \cup m_2(F')\,.$$

We can realize M' as the quotient space $P/{\approx}$, where the equivalence relation $\approx$ is generated by $z \approx m_1(z)$ and $z \approx m_2(z)$ for all $z \in F'$.

Let $Q = \mathrm{int}(P)$ be the interior of P in $\mathbb{H}$, i.e., the interior of the ideal hyperbolic polygon $A\infty BCE$ minus the point D. As an open subset of $\mathbb{H}$, the surface Q inherits a hyperbolic structure. We can also realize M' as $Q/{\approx}$, where $\approx$ denotes the restriction of the given equivalence relation on P to Q. Each point of M' then has a neighborhood U that is homeomorphic to one, two or three neighborhoods V_i in Q, and the coordinate transformations between these neighborhoods V_i and V_j are given by m_1, m_2, their inverses and composites of these, i.e., by hyperbolic isometries. Hence the hyperbolic structure on Q descends to a hyperbolic structure on M'.

Problem 3

3a.

$$\begin{aligned}
x_u &= (\cos v, \sin v, 0)\\
x_v &= (-u\sin v, u\cos v, 1)\\
E &= \cos^2 v + \sin^2 v + 0^2 = 1\\
F &= (\cos v)(-u\sin v) + (\sin v)(u\cos v) + (0)(1) = 0\\
G &= (-u\sin v)^2 + (u\cos v)^2 + 1^2 = 1 + u^2\\
x_u \times x_v &= (\sin v, -\cos v, (\cos v)(u\cos v) - (\sin v)(-u\sin v)) = (\sin v, -\cos v, u)\\
\|x_u \times x_v\| &= \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}\\
N &= \frac{(\sin v, -\cos v, u)}{\sqrt{1+u^2}}\,.
\end{aligned}$$

3b.

$$\begin{aligned}
x_{uu} &= (0,0,0)\\
x_{uv} &= (-\sin v, \cos v, 0)\\
x_{vv} &= (-u\cos v, -u\sin v, 0)\\
e &= 0\\
f &= \frac{-\sin^2 v - \cos^2 v + u\cdot 0}{\sqrt{1+u^2}} = -\frac{1}{\sqrt{1+u^2}}\\
g &= \frac{(\sin v)(-u\cos v) - (\cos v)(-u\sin v) + u\cdot 0}{\sqrt{1+u^2}} = 0\\
K &= \frac{(0\cdot 0 - (-1/\sqrt{1+u^2})^2)}{(1+u^2-0^2)} = \frac{-1/(1+u^2)}{1+u^2} = -\frac{1}{(1+u^2)^2}\,.
\end{aligned}$$

3c. We calculate

$$\alpha'(s) = (-\frac{1}{\sqrt{2}}\sin(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}}\cos(\frac{s}{\sqrt{2}}), \frac{1}{\sqrt{2}})$$

and

$$\alpha''(s) = (-\frac{1}{2}\cos(\frac{s}{\sqrt{2}}), -\frac{1}{2}\sin(\frac{s}{\sqrt{2}}), 0)\,.$$

We have

$$\|\alpha'(s)\| = \frac{1}{2}\sin^2(\frac{s}{\sqrt{2}}) + \frac{1}{2}\cos^2(\frac{s}{\sqrt{2}}) + \frac{1}{2} = 1\,,$$

so $\alpha\colon [0, 2\pi\sqrt{2}] \to S$ is traversed at unit speed and parametrized by arc length.

The tangent plane T_pS of S at $p = \alpha(s) = x(1, s/\sqrt{2})$ contains the tangent vectors

$$x_u(1, s/\sqrt{2}) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), 0)$$

and

$$x_v(1, s/\sqrt{2}) = (-\sin(s/\sqrt{2}), \cos(s/\sqrt{2}), 1)\,.$$

These are orthogonal. The unit tangent vector $T(s) = \alpha'(s)$ of α at p is parallel to the tangent vector $x_v(1, s/\sqrt{2})$. Hence the unit bitangent vector $B(s)$ is parallel to $x_u(1, s/\sqrt{2}) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), 0)$. This is a unit vector pointing out of R, so $B(s)$ is its negative:

$$B(s) = (-\cos(s/\sqrt{2}), -\sin(s/\sqrt{2}), 0)\,.$$

Hence the geodesic curvature is

$$k_g(s) = -\cos(s/\sqrt{2})(-\frac{1}{2}\cos(s/\sqrt{2})) - \sin(s/\sqrt{2})(-\frac{1}{2}\sin(s/\sqrt{2})) + 0\cdot 0 = \frac{1}{2}$$

for all values of s.

3d.

$$\iint_R K\,dA + \int_{\partial R} k_g\,ds + \sum_k \epsilon_k = 2\pi\chi(R)\,.$$

The surface integral of the Gaussian curvature is

$$\iint_R K\,dA = \int_0^{2\pi}\int_0^1 \frac{-1}{(1+u^2)^2}\|x_u \times x_v\|\,du\,dv = 2\pi\int_0^1 \frac{-1}{(1+u^2)^{3/2}}\,du = -\pi\sqrt{2}\,.$$

The line integral of the geodesic curvature along the image of α is

$$\int_\alpha k_g\,ds = \int_0^{2\pi\sqrt{2}} \frac{1}{2}\,ds = \pi\sqrt{2}\,.$$

The three other parts of ∂R are geodesics, hence $k_g = 0$ along these curves. Thus

$$\int_{\partial R} k_g\,ds = \pi\sqrt{2} + 0 + 0 + 0 = \pi\sqrt{2}\,.$$

The angular change of direction at each corner A, B, C and D is $\epsilon_k = \pi/2$. For example, at A, the tangent to the curve α is $\alpha'(0) = (0, 1, 0)$ and the tangent to the line segment $[D, A]$ is $(1, 0, 0)$. Hence the interior angle at A is $\pi/2$ and the change of direction is $\pi - \pi/2 = \pi/2$. The other three cases are very similar. Hence

$$\sum_k \epsilon_k = 4\cdot\pi/2 = 2\pi\,.$$

The Euler characteristic of R is 1, e.g. because $R \cong [0,1]\times[0,2\pi]$ can be triangulated with $v = 4$ vertices, $e = 5$ edges and $f = 2$ faces, so $\chi(R) = 4 - 5 + 2 = 1$. Hence

$$2\pi\chi(R) = 2\pi\,.$$

The Gauss-Bonnet formula asserts that

$$-\pi\sqrt{2} + \pi\sqrt{2} + 2\pi = 2\pi\,,$$

which is correct.