UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT4510 — Geometrical structures.

Day of examination: Wednesday, December 15, 2010.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each item (1, 2a, 2b etc.) counts 10 points.

Problem 1

Let $f \in \text{Möb}^+(\mathbb{H})$ be defined by

$$f(z) = \frac{\sqrt{2}z + 1}{z + \sqrt{2}}.$$

Decide wether f is parabolic, hyperbolic or elliptic. Write f explicitly as a conjugate of a map of normal form.

Problem 2

Let S be a Riemannian surface, $\mathbf{x} = \mathbf{x}(u, v) : U \to S$ a parametrization, and $\beta(s) = \mathbf{x}(u(s), v(s))$ a smooth curve on S, parametrized by arc-length. The covariant second derivative $D\beta''(s)$ is defined as:

$$\begin{aligned} D\beta''(s) = {} & (u'' + (u')^2\Gamma^1_{11} + 2u'v'\Gamma^1_{12} + (v')^2\Gamma^1_{22})\mathbf{x}_u \\ & + (v'' + (u')^2\Gamma^2_{11} + 2u'v'\Gamma^2_{12} + (v')^2\Gamma^2_{22})\mathbf{x}_v, \end{aligned}$$

where the Γ^k_{ij} are given by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v & F_v - \frac{1}{2}G_u \\ F_u - \frac{1}{2}E_v & \frac{1}{2}G_u & \frac{1}{2}G_v \end{bmatrix}.$$

a) Let $S = \mathbb{H}$ with hyperbolic Riemannian metric, and let $\mathbf{x}(x, y) = x + yi$. Find the expression $D\beta''(s)$ for a curve $\beta(s) = x(s) + y(s)i$ in $\mathbb{H}$, parametrized by arc-length.

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b) Let C_{y_0} be the line in $\mathbb{H}$, given by $C_{y_0} = \{z = x + yi \,|\, y = y_0\}$. Explain why the geodesic curvature k_g of C_{y_0} is constant along C_{y_0}. Compute k_g, when the unit normal vector along C_{y_0} is chosen in the same direction as the imaginary axis.

c) Let R be the bounded region in $\mathbb{H}$ bounded by the circle $\{z \,|\, |z| = 1\}$ and the line $C_{\frac{1}{2}\sqrt{2}}$. Calculate the hyperbolic area of R, by using the Gauss-Bonnet theorem.

d) Calculate the hyperbolic area of R in c), directly by integration (without using Gauss-Bonnet).

Problem 3

Consider a hyperbolic triangle T in $\mathbb{D}$ with vertices $-r, ri, r$, $r \in (0, 1)$. Find r such that the sum of the angles in T is equal $\frac{2\pi}{3}$.
(Here you may use the second law of cosines for a hyperbolic triangle which is given by

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a.)$$

Problem 4

Consider the surface

$$S = \{(x, y, z) \,|\, \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1\} \subset \mathbb{R}^3,$$

where $a, b > 0$. A parametrization of this surface is given by

$$x = a\cos u\cos v, y = a\cos u\sin v, z = b\sin u.$$

a) Compute the metric

$$ds^2 = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2$$

in these coordinates.

b) Compute the Gaussian curvature for S at a point $(x(u, v), y(u, v), z(u, v))$.

c) Show , by using the Gauss-Bonnet theorem, that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{ab^2\cos u du}{(a^2\sin^2 u + b^2\cos^2 u)^{\frac{3}{2}}} = 2$$

for any a, b>0.

d) Let $\mathbf{x}(u, v) = (g(u)\cos v, g(u)\sin v, h(u))$, $g(u) > 0$ be a parametrization of a regular surface of rotation. Show that the curves v =constant are always geodesics. When is a curve u =constant a geodesic ?

THE END