UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT4510 — Geometric structures.

Day of examination: Friday, December 14, 2012.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each item (1a, 1b, 2a, etc.) counts 10 points.

Problem 1

a) Let Q be a regular hyperbolic quadrilateral (a hyperbolic quadrilateral with all four sides of equal hyperbolic length and all four angles of equal measure) with angle equal α. Explain why the hyperbolic area of Q is equal $2\pi - 4\alpha$. Assume $\alpha = \frac{\pi}{3}$. Find the hyperbolic length of the side of Q in this case.

(Here the second law of cosine

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a,$$

may be useful, also recall that if $y = \cosh x$, $x \geq 0$ then $x = \ln(y + \sqrt{y^2 - 1}$.)

b) Let T be a simply asymptotic triangle in $\mathbb{D}$ or $\mathbb{H}$, with the angles at the two vertices in $\mathbb{D}$ or $\mathbb{H}$ equal $\frac{\pi}{4}$ and $\frac{\pi}{2}$ respectively. Find the hyperbolic length of the finite segment between these two vertices.

(Here you may use that the second law of cosine is valid for simply asymptotic trangles.)

Problem 2

a) Let $h \in \text{Möb}^+(\mathbb{H})$ be defined by

$$h(z) = \frac{4z}{3z + 1}.$$

Decide wether f is parabolic, hyperbolic or elliptic. Write f explicitly as a conjugate of a map on normal form.

(Continued on page 2.)

b) Let $f \in \text{Möb}^-(\mathbb{H})$ be defined by

$$f(z) = \frac{4\bar{z}}{5\bar{z} - 1}.$$

Find the fixpoints of f. Find an inversion $g \in \text{Möb}^-(\mathbb{H})$ and a hyperbolic transformation $k \in \text{Möb}^+(\mathbb{H})$ such that $f = g \circ k$ and $g \circ k = k \circ g$. g is an inversion in a $\mathbb{H}$-line ℓ, find ℓ.

Problem 3

a) Let $\Omega(Y)$ be the bounded region in $\mathbb{H}$, bounded by the curve $xy = 1$, and the lines $y = Y$, $x = 1$, where $Y > 1$ is a constant and $z = x + yi \in \mathbb{H}$. Calculate the hyperbolic area $A((\Omega(Y))$ of $\Omega(Y)$ and find $\lim_{Y \to \infty} A(\Omega(Y))$.

b) Let

$$\alpha(t) = (\frac{1}{2}t\sqrt{1-t^2} + \frac{\arcsin t}{2}) + \frac{1}{2}t^2 i, \; t \in [\frac{1}{2}, 1]$$

be a curve in $\mathbb{H}$. Calculate the hyperbolic arc-length of α. (Recall that $(\arcsin t)' = \frac{1}{\sqrt{1-t^2}}$.)

Problem 4

a) A regular surface $S \subset \mathbb{R}^3$ is parametrized by

$$\mathbf{x}(u,v) = ((\frac{1}{4}v^4 - \frac{1}{2}v^2 + \frac{1}{2})\cos u, (\frac{1}{4}v^4 - \frac{1}{2}v^2 + \frac{1}{2})\sin u, v), \; u, v \in \mathbb{R}.$$

Find the first fundamental form of S with respect to this parametrization and find the Gaussian curvature of S.

b) Show that the curves given by $v = -1, v = 0$ and $v = 1$ are geodesics on S.

c) Let α be the curve on S given by $v = 2$. Calculate the geodesic curvature of α , where we choose the normal vector $n_\alpha(s)$ in $T_{\alpha(s)}S$ with positive z- component.

d) Let

$$R = \{\mathbf{x}(u,v) \in S \,|\, u \in \mathbb{R}, \, -1 \leq v \leq 1\}.$$

Use The Gauss-Bonnet Theorem to show that $\int\int_R K dA = 0$.

THE END