UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4510 — Geometric structures

Day of examination: December 14th 2015

Examination hours: 09.00 – 13.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each of the eleven subproblems (1a, 1b, 2a, etc.) carries equal weight. Please remember to justify all your answers.

Problem 1

Let x_1, x_2 and $x_3 \in \mathbb{R}$ be three pairwise distinct real numbers. The fractional linear transformation

$$m(z) = \frac{z - x_2}{z - x_3} \cdot \frac{x_1 - x_3}{x_1 - x_2}$$

satisfies $m(x_1) = 1$, $m(x_2) = 0$ and $m(x_3) = \infty$. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half-plane.

1a

Consider the special case $x_1 = -1$, $x_2 = 0$ and $x_3 = +1$. Calculate $m(i)$, and explain why $m(\mathbb{H}) \neq \mathbb{H}$.

1b

Returning to the general case, find a condition on x_1, x_2 and x_3, expressed in terms of the product

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\,,$$

such that $m(\mathbb{H}) = \mathbb{H}$ if and only if the condition holds. You may use the fact that $m\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is invertible, and that it maps $\bar{\mathbb{C}}$-circles onto $\bar{\mathbb{C}}$-circles.

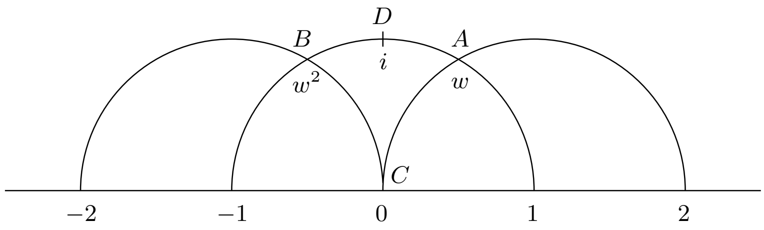
Problem 2

Let

$$w = \cos(\pi/3) + i\sin(\pi/3) = (1 + i\sqrt{3})/2\,.$$

Note that $w^2 = w - 1 = (-1 + i\sqrt{3})/2$. Let $\triangle ABC$ be the ideal (= asymptotic) hyperbolic triangle with vertices $A = w$, $B = w^2$ and $C = 0$. It

(Continued on page 2.)



lies in the closure in $\bar{\mathbb{C}}$ of the upper half-plane model $\mathbb{H}$ for the hyperbolic plane.

2a

What is the hyperbolic area of $\triangle ABC$?

2b

The fractional linear transformation

$$m_1(z) = -1/z$$

satisfies $m_1(\mathbb{H}) = \mathbb{H}$. Is m_1 of hyperbolic, parabolic or elliptic type? Let $D = i$ be the midpoint of the hyperbolic line segment $[A, B]$. What is the image of the hyperbolic line segment $[A, D]$ under m_1? What is the image of $\triangle ABC$ under m_1?

2c

The fractional linear transformation

$$m_2(z) = z/(z + 1)$$

satisfies $m_2(\mathbb{H}) = \mathbb{H}$. Is m_2 of hyperbolic, parabolic or elliptic type? What is the image of the hyperbolic ray $\overrightarrow{BC}$ under m_2? What is the image of $\triangle ABC$ under m_2?

2d

Let $F = \square ADBC$ be the asymptotic hyperbolic quadrangle with vertices A, D, B and C. (As a set, it is equal to $\triangle ABC$.) Let $M = F/\!\sim$ be the quotient space where each boundary point $z \in [A, D]$ is identified with its image $m_1(z)$, and each boundary point $z \in \overrightarrow{BC}$ is identified with its image $m_2(z)$.

The space M is homeomorphic to exactly one of the standard closed surfaces $M_g = S(g, 0)$ for $g \geq 0$ and $N_h = S(0, h)$ for $h \geq 1$. Which one?

2e

Does the closed surface M admit a hyperbolic structure?

Let M' be the complement of the image of $\{A, B, C, D\} \subset F$ under the canonical map $F \to M$. Does the surface M' admit a hyperbolic structure?

(Continued on page 3.)

Problem 3

The *helicoid* is a regular surface S in $\mathbb{R}^3$, parametrized by

$$x(u, v) = (u \cos v, u \sin v, v)$$

for $(u, v) \in \mathbb{R}^2$. Let $R \subset S$ be the compact region parametrized by

$$x^{-1}(R) = [0, 1] \times [0, 2\pi]\,.$$

The boundary ∂R is the union of four regular curves, namely the curve from $A = x(1, 0) = (1, 0, 0)$ to $B = x(1, 2\pi) = (1, 0, 2\pi)$ parametrized by

$$\alpha(s) = x(1, s/\sqrt{2}) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2})$$

for $s \in [0, 2\pi\sqrt{2}]$, and the three Euclidean line segments $[B, C]$, $[C, D]$ and $[D, A]$, where $C = x(0, 2\pi) = (0, 0, 2\pi)$ and $D = x(0, 0) = (0, 0, 0)$.

3a

Calculate the tangent vectors x_u and x_v, the first fundamental form $E = x_u \cdot x_u$, $F = x_u \cdot x_v$ and $G = x_v \cdot x_v$, the cross product $x_u \times x_v$ and the unit normal vector $N = (x_u \times x_v)/\|x_u \times x_v\|$.

3b

Calculate the second order partial derivatives x_{uu}, x_{uv} and x_{vv}, the second fundamental form $e = N \cdot x_{uu}$, $f = N \cdot x_{uv}$ and $g = N \cdot x_{vv}$, and the Gaussian curvature $K = (eg - f^2)/(EG - F^2)$.

3c

Calculate $\alpha'(s)$ and $\alpha''(s)$. Verify that α is parametrized by arc length. Calculate the geodesic curvature $k_g(s) = B(s) \cdot \alpha''(s)$ of α, where $B(s) = n_\alpha(s)$ is the unit bitangent vector pointing into R at $\alpha(s)$.

3d

State the Gauss-Bonnet formula for $R \subset S$, as a relation between the four terms

$$\iint_R K\, dA \quad , \quad \int_{\partial R} k_g\, ds \quad , \quad \sum_k \epsilon_k \quad \text{and} \quad 2\pi\chi(R)\,.$$

Calculate each of these four terms directly, and verify that the formula is correct in this case. You may use that

$$\int_0^1 \frac{du}{(1 + u^2)^{3/2}} = \left[\frac{u}{\sqrt{1 + u^2}}\right]_0^1 = \frac{\sqrt{2}}{2}\,.$$

THE END