Last time

* An isometry $f:(S,\langle,\rangle) \to (\tilde{S}, \tilde{\langle,\rangle})$ is a diffeo which preserves the Riemannian structure

$$\left(\|df_p(v)\|_{f(p)} = \|v\|_p \quad \forall p \in S,\ \forall v \in T_pS \right)$$

* An intrinsic property is a property which is preserved by all isometries $\rightsquigarrow$ does not depend on the representation of S. Ex: Arc-lengt, area, E, F, G

* An extrinsic property is a property that do depend on the representation of S.

Ex Normal v.f. of $S \subset \mathbb{R}^3$

* The Gaussian curvature of $S \subset \mathbb{R}^3$ at p

$$K(p) = \det(dN_p), \quad dN_p : T_pS \to \underbrace{T_pS^2}_{= T_pS}, \quad \underbrace{N: S \to S^2}_{\text{unit normal}}$$

In Local coord: $K_p = \dfrac{eg - f^2}{EG - F^2}$, where

$e = N \cdot x_{uu}$, $f = N \cdot x_{uv}$, $g = N \cdot x_{vv}$ extrinsic

$\Rightarrow$ Gaussian curvature seems extrinsic

Today Will prove it is intrinsic $\rightsquigarrow$ remarkable!

+ geometric interpretation of Gaussian curvature

+ start studying Geodesics = "lines" in Riemannian geometry

<u>Def</u> $e\,du^2 + 2f\,du\,dv + g\,dv^2$ the

<u>second fundamental form</u> of S

⚠ Not intrinsic, but gives important info on how

S lies in $\mathbb{R}^3$

Geometric interpretation of Gaussian curvature?

<u>Ex 5.5.5</u> Let $S \subset \mathbb{R}^3$ graph of smooth $h: \mathcal{O} \to \mathbb{R}$, $\mathcal{O} \subset \mathbb{R}^2$
$(x,y) \mapsto h(x,y)$

$\Rightarrow S$ param by $z(x,y) = (x, y, h(x,y))$, $z_x = (1, 0, h_x)$, $z_y = (0, 1, h_y)$

$\Rightarrow E(x,y) = 1 + h_x^2(x,y)$, $G(x,y) = 1 + h_y^2(x,y)$, $F(x,y) = h_x(x,y) \cdot h_y(x,y)$

$$\Rightarrow * \quad N(x,y) = \frac{z_x \times z_y}{\|z_x \times z_y\|} = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}}$$

$* \quad z_{xx} = (0, 0, h_{xx}),\ z_{xy} = (0, 0, h_{xy}),\ z_{yy} = (0, 0, h_{yy})$

$$\Rightarrow e(x,y) = [N \cdot z_{xx}] = \frac{h_{xx}(x,y)}{\sqrt{1+h_x^2+h_y^2}},\quad f(x,y) = \frac{h_{xy}(x,y)}{\sqrt{1+h_x^2+h_y^2}},\quad g(x,y) = \frac{h_{yy}(x,y)}{\sqrt{1+h_x^2+h_y^2}}$$

$$\Rightarrow K(x,y) = \frac{h_{xx}(x,y)h_{yy}(x,y) - h_{xy}^2(x,y)}{\underbrace{\left((1+h_x^2)(1+h_y^2) - (h_x h_y)^2\right)}_{= 1 + h_x^2 + h_y^2}(1+h_x^2+h_y^2)} = \frac{\det H(h)}{(1+h_x^2+h_y^2)^2}$$

where $H(h)$ is the <u>Hessian</u> of h

$$\parallel$$

$$\begin{bmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{bmatrix}$$

<u>Recall</u> At a non-degenerate crit pt p of h $\left(\begin{array}{l}\text{meaning} \\ h'(p)=0, \\ \det H(h)\neq 0\end{array}\right)$

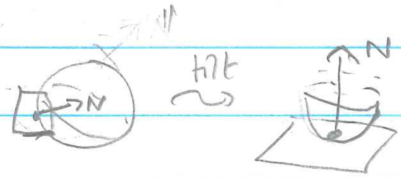
we have that

$\det(H(h)) > 0 \Rightarrow p$ max or min

$\det(H(h)) < 0 \Rightarrow p$ saddle

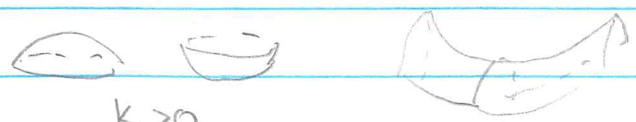
<u>Prp 5.1.9</u> A regular surface $S \subset \mathbb{R}^3$ coincides w/ a graph of a smooth fcn in a nbhd of every point and the point can be choosen as a critical point for the fcn (after tilting)

<u>Pf</u> Implicit fcn thm



$\{(\underbrace{u,v}_{\in T_pS}, g(u,v)N)\}$ □

$\rightsquigarrow$ geometric interpretation of the <u>sign</u> of the curvature:



$K>0$ (max/min)

saddle

$K<0$

<u>Rmk</u> A <u>cpt</u> regular surface cannot [illegible] everywhere negative curvature

<u>Pf</u> (cpt $\Rightarrow$ bdd $\Rightarrow$ if we project to any line [illegible] we must get a max & min [illegible] >0. □

One consequence:

<u>Prop 5.5.6</u> A compact surface $S \subset \mathbb{R}^3$ must have a point where the curvature is > 0.

<u>Pf</u> Consider $f: S \to \mathbb{R}$, $f(q) = |q|^2 = q \cdot q$

(squared distance from 0)

S cpct $\Rightarrow \exists$ max point p for f, clearly $p \neq 0$

<u>Algebraic Lma</u> Let A a real 2×2-matrix s.t.

A self-adjoint w.r.t $\langle\, , \rangle$

$\langle Av, w \rangle = \langle v, Aw \rangle \quad \forall v, w \in \mathbb{R}^2$. Then $\det A > 0$

$\Leftrightarrow \langle A(v), v \rangle \neq 0 \quad \forall v \in \mathbb{R}^2 \setminus 0$.

<u>Pf</u> Exercise

Exercise: dN_p self-adjoint w.r.t. $\langle\, , \rangle_p$

$\Rightarrow$ To prove the proposition, it remains to prove that

$$\underbrace{\langle dN_p(v), v \rangle}_{= dN_p(v) \cdot v} > 0 \quad \forall v \in T_pN,\ v \neq 0$$

So let $v \in T_pS$, $v \neq 0$ be given, let w curve on S

s.t. $w(0) = p$, $w'(0) = v$ & consider $N(w(t))$

Know $N(w(t)) \cdot w'(t) = 0$ $\forall t$ since $N(w(t))$ normal to $T_{w(t)}S$

$\Rightarrow 0 = \frac{d}{dt}\left(N(w(t)) \cdot w'(t)\right) = (N \circ w)'(t) \cdot w'(t) + N(w(t)) \cdot w''(t)$

at $t=0$: $-N(p) \cdot w''(0) = \underbrace{(N \circ w)'(0)}_{dN_p(w'(0))} \cdot w'(0) = dN_p(v) \cdot v$ $(*)$

Now consider $g(t) = f(w(t)) = w(t) \cdot w(t)$

$t=0$ gives max for $g(t)$ $\Rightarrow$ $g'(0) = 0$, $g''(0) \leq 0$

But $g'(t) = 2\, w'(t) \cdot w(t)$, $g''(t) = 2\, w'(t) \cdot w'(t) + 2\, w''(t) \cdot w(t)$

$\Rightarrow$ $\underbrace{2\, v \cdot p}_{=g'(0)} = 0$, $\underbrace{2|v|^2 + 2p \cdot w''(0)}_{=g''(0)} \leq 0$ $(**)$

this holds $\forall v \in T_pS$ $\Rightarrow$ p normal to T_pS $\Rightarrow$

$N(p) = \frac{p}{\|p\|}$

Hence $dN_p(v) \cdot v \underset{(*)}{=} -w''(0) \cdot N(p) = -w''(0) \cdot \frac{p}{|p|} \underset{(**)}{\geq} \frac{|v|^2}{|p|} > 0$

$\square$

Now it's time to prove

F(17): 4

Theorema egregium

Gaussian curvature is intrinsic

Pf $K = \dfrac{\overbrace{(x_{uu}\cdot N)}^{e}\overbrace{(x_{vv}\cdot N)}^{g} - \overbrace{(x_{uv}\cdot N)}^{f}{}^2}{\underbrace{EG-F^2}_{\text{intrinsic}}}$ ← express this in an intrinsic way

Use $* \; N = \dfrac{x_u \times x_v}{\|x_u \times x_v\|} = \dfrac{x_u\times x_v}{\sqrt{EG-F^2}}$

$* \; (a\cdot(b\times c)) = \det\begin{pmatrix} -a- \\ -b- \\ -c- \end{pmatrix} = \det\left(\begin{pmatrix} -a- \\ -b- \\ -c- \end{pmatrix}^t\right)$

$$\Rightarrow K(EG-F^2)^2 = (x_{uu}\cdot(x_u\times x_v))(x_{vv}\cdot(x_u\times x_v)) - (x_{uv}\cdot(x_u\times x_v))^2$$

$$= \det\begin{bmatrix} -x_{uu}- \\ -x_u- \\ -x_v- \end{bmatrix}\cdot\det\begin{bmatrix} -x_{vv}- \\ -x_u- \\ -x_v- \end{bmatrix}^t - \det\begin{bmatrix} -x_{uv}- \\ -x_u- \\ -x_v- \end{bmatrix}\cdot\det\begin{bmatrix} -x_{uv}- \\ -x_u- \\ -x_v- \end{bmatrix}^t$$

$$= \det\begin{bmatrix} x_{uu}\cdot x_{vv} & x_{uu}\cdot x_u & x_{uu}\cdot x_v \\ x_u\cdot x_{vv} & E & F \\ x_v\cdot x_{vv} & F & G \end{bmatrix} - \det\begin{bmatrix} x_{uv}\cdot x_{uv} & x_{uv}\cdot x_u & x_{uv}\cdot x_v \\ x_u\cdot x_{uv} & E & F \\ x_v\cdot x_{uv} & F & G \end{bmatrix}$$

$$= \det\begin{bmatrix} x_{uu}\cdot x_{vv} - x_{uv}\cdot x_{uv} & & \\ & & \\ & & \end{bmatrix} - \det\begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix}$$

Now use $E_u = 2x_{uu}\cdot x_u$, $F_u = x_{uu}\cdot x_v + x_u\cdot x_{uv}$, $G_u = 2x_{uv}\cdot x_v$

$E_v = 2x_{uv}\cdot x_u$, $F_v = x_{uv}\cdot x_v + x_u\cdot x_{vv}$, $G_v = 2x_{vv}\cdot x_v$

$x_{uu}\cdot x_{vv} - x_{uv}\cdot x_{uv} = (x_u\cdot x_{vv})_u - (x_u\cdot x_{uv})_v$ □

$+ E_u, E_v, G_u, G_v, F_u, F_v$ intrinsic.

Consequences

* $\mathbb{R}^2$ & S^2 not locally isometric

Q Why?

* If the Riemannian surface S has a gp of isometries acting transitively ($\forall p,q \in S$ $\exists$ isometry $f: S \to S$ s.t. $f(q)=p$) then S has constant curvature (ex $\mathbb{R}^2, S^2, \mathbb{H}$)

Rmk $\exists$ cpct surfaces S s.t. $S \not\subset \mathbb{R}^3$ regular

Q ex?

Can still define curvature at $p \in S$:

* If $p \in U \overset{\subset S}{} \xrightarrow{f} S' \subset \mathbb{R}^3$ isometry of nbhd then define $K_S(p) = K_{S'}(f(p))$. Theorema egregium $\Rightarrow$ independent of choice of f & S', hence well-def.

* If above not possible, then can use Riemannian connections to define Gaussian curvature (can do that for any Riemannian surface). Beyond the scope of this course.