Last time: Derived formulas for arc-length & area in $\mathbb{H}$:

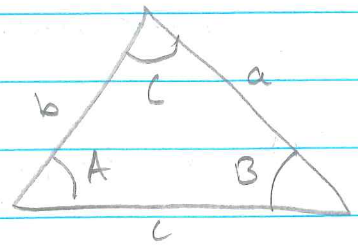
- $ds^2 = \dfrac{dx^2+dy^2}{y^2} \Rightarrow s(C) = \displaystyle\int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}{y}\,dt$

Where C param. by $z(t) = (x(t), y(t))$, $t \in [a,b]$.

- $A_{\mathbb{H}}(\Omega) = \displaystyle\iint_\Omega \frac{dx\,dy}{y^2}$

Today: §2.9 Trigonometry in the hyperbolic plane

Derive formulas sim. to the Law of sines/cosines



in $\mathbb{E}^2$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\Rightarrow$ · can calculate side c if know B, C & side b

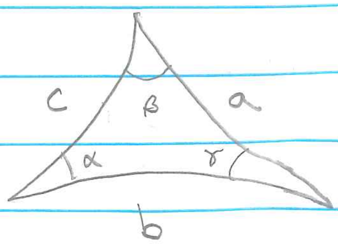
· or angle B if know side b, c and angle B

In hyperbolic geometry 2 types of triangles:

finite triangles: w/ vertices in $\mathbb{D}/\mathbb{H}$

asymptotic triangles: w/ 1 or more vertices in $\partial\mathbb{D}/\overline{\mathbb{R}}$

Start w[illegible] finite : • sides = segments



α, β, γ both angles & angle measures

a, b, c both segments & hyperbolic length of the segments.

Prp 2.9.1 (The first Law of cosines)

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$$

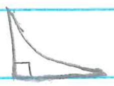
$$\cosh b = \cosh c \cosh a - \sinh c \sinh a \cos \beta$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cosh \gamma$$

[Euclidean $c^2 = a^2 + b^2 - 2ab \cos \gamma$; again use Taylor's formula for small triangles]
$\cosh x \approx 1 + \frac{x^2}{2}$, $\sinh x \approx x$

Cor 2.9.2 (The hyperbolic Pythagorean thm)

If $\alpha = \pi/2$, then $\cosh a = \cosh b \cosh c$



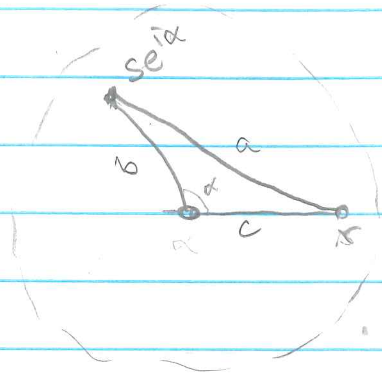
$$\underset{\left(1 + \frac{a^2}{2} + O(a^4)\right)}{\cosh a} = \underset{\left(1 + \frac{b^2}{2} + O(b^4)\right)}{\cosh b} \; \underset{\left(1 + \frac{c^2}{2} + O(c^4)\right)}{\cosh c}$$

$\Rightarrow$ $a^2 = b^2 + c^2 +$ (terms of order at least 4)

$\Rightarrow$ for small triangles we get $\approx$ the Pythagorean thm

Pf of Prp 2.9.1

Prove this in $\mathbb{D}$, assuming the triangle is in "standard position": • translate so α vertex at 0



• rotate so β vertex at $r\in(0,1)$

• reflect so γ vertex at

z w/ $\operatorname{Im} z > 0 \Rightarrow$

$z = se^{i\alpha}$, $s \in (0,1)$, $\alpha \in (0,\pi)$

$$\Rightarrow \cosh a = \cosh(d_{\mathbb{D}}(r, se^{i\alpha})) = 1 + 2\frac{|r-se^{i\alpha}|^2}{(1-r^2)(1-s^2)}$$

$$\boxed{|r-se^{i\alpha}|^2 = (r-s\cos\alpha)^2 + (s\sin\alpha)^2 = r^2 - 2rs\cos\alpha + s^2}$$

$$= \frac{1+r^2}{1-r^2}\,\frac{1+s^2}{1-s^2} - \frac{2r}{1-r^2}\,\frac{2s}{1-s^2}\cos\alpha$$

$r = \tanh(c/2)$, $s = \tanh(b/2)$

$$\Rightarrow \frac{1+r^2}{1-r^2} = \frac{1+\tanh^2(c/2)}{1-\tanh^2(c/2)} = \cosh c, \quad \frac{2r}{1-r^2} = \frac{2\tanh(c/2)}{1-\tanh^2(c/2)} = \sinh c$$

$$\frac{1+s^2}{1-s^2} = \cosh b \qquad \frac{2s}{1-s^2} = \sinh b$$

$$\Rightarrow \cosh a = \cosh c \cosh b - \sinh c \sinh b \cos\alpha$$

+ use symmetry to get the other 2 formulas. $\square$

From this we can then derive:

Prp 2.9.3 (The hyperbolic Law of Sines)

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$$

Rmk $\sinh x \approx x$ if x small $\Rightarrow$ for small triangles this is the Euclidean law of sines.

Prp 2.9.4 (The second Law of cosines)

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cosh a$$

$$\cos\beta = -\cos\gamma\cos\alpha + \sin\gamma\sin\alpha\cosh b$$

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c.$$

Pf of Prp 2.9.3 & Prp 2.9.4

Manipulation of the formulas in the first Law of cosines, see book. $\square$

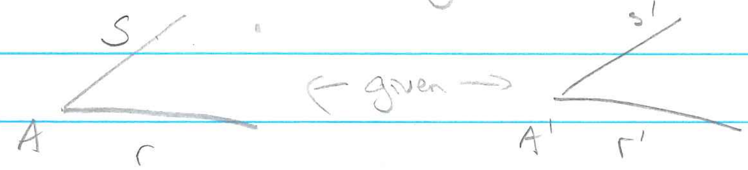
Consequences

First law of cosine $\Rightarrow$ SSS (side - side - side)

the lengths of all three sides determine all the angles (same in $\mathbb{E}^2$)

$\leadsto$ Prp 2.9.6 Congruence of angles can be characterized in terms of congruence of segments.

Pf



Choose vertices $B \in r$, $C \in S$, $B, C \neq A$

Axiom C1 $\Rightarrow \exists\, B' \in r', C' \in s'$ s.t. $[A', B'] \cong [A, B]$
$[A', C'] \cong [A, C]$

Then SSS $\Rightarrow \angle(r, s) \cong \angle(r', s') \Longleftrightarrow [B', C'] \cong [B, C]$.

$\square$

Second law of cosine $\Rightarrow$ AAA (angle-angle-angle)

- if we know all angles in a triangle, then we also know the sides & hence the whole triangle, up to congruence

⚠ Not the case in Euclidean geometry.

Law of sines A-S-A (angle-side-angle)

In both hyperbolic & euclidean : 2 angles + side

between them determines third angle + rest of the sides

$\Rightarrow$ the whole triangle up to congruence.

Note In Euclidean geom, two angles of a triangle determine

the third [since sum $= \pi$].

⚠ Not the case in hyperbolic geometry!

If $\alpha+\beta+\gamma=\pi$, then $\alpha = \pi-\beta-\gamma$ and

$\cos\alpha = \cos(\pi-(\beta+\gamma)) = -\cos(\beta+\gamma) = -\cos\beta\cos\gamma + \sin\beta\sin\gamma$

But the Second law of cosines $\Rightarrow$

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\underbrace{\cosh a}_{>1}$$

$\Rightarrow$ in a hyperbolic triangle $\alpha+\beta+\gamma \neq \pi$

In fact $\pi > \alpha+\beta+\gamma$, which follows from

Prp 2.8.5 The hyperbolic area of a triangle w/ angles α, β, γ is given by $\pi - (\alpha+\beta+\gamma)$.

[and $0 < A_{\mathbb{H}}(\text{triangle}) = \pi - (\alpha+\beta+\gamma) \Rightarrow (\alpha+\beta+\gamma) < \pi$]

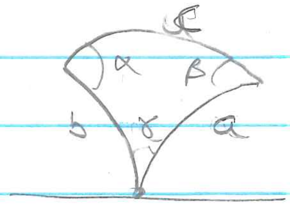
This proposition also holds for asymptotic triangles w/ 1 or more ideal vertices: a vertex in $\partial\overline{\mathbb{D}} / \overline{\mathbb{R}}$.

By def. the angle measure of an ideal vertex $= 0$

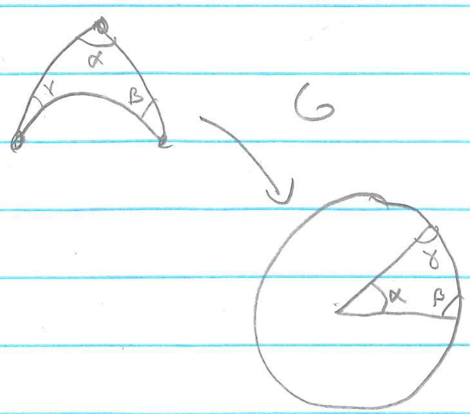
3 types of asympt. triangles:

(i) Simply asymptotic: Have 1 ideal vertex $\Rightarrow$ sides consists of 2 rays & 1 segment

$\Rightarrow$ length of a & b $= \infty$, $\gamma = 0$ $\mathbb{H} \supset$

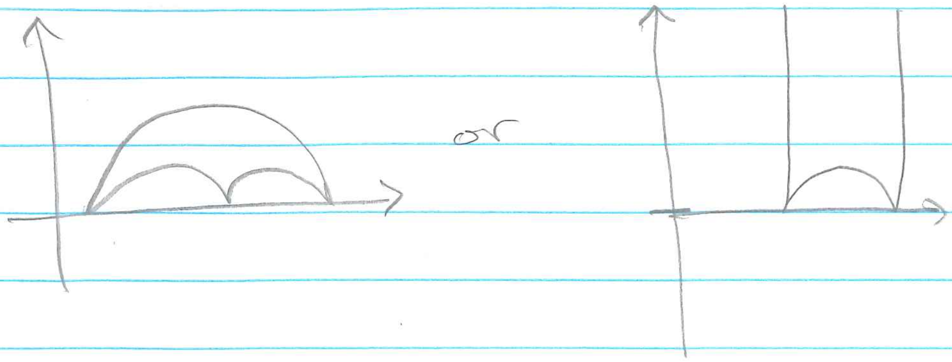


(ii) Doubly asymptotic: 2 ideal vertices, sides: 2 rays & 1 hyperbolic line



$\rightsquigarrow$ α determines the triangle up to congruence.

iii) Triply asymptotic 3 ideal vertices

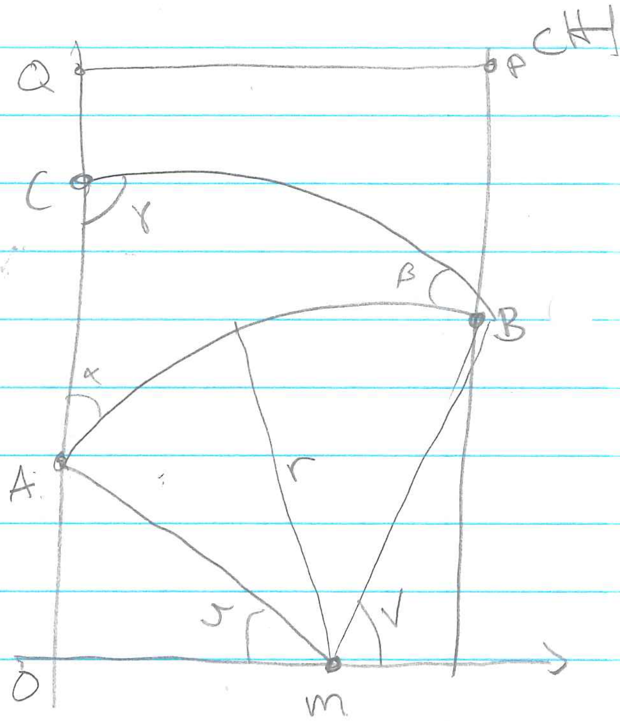


all sides are hyperbolic lines

Exc 2.2.7 Any triple of pts $\subset \bar{\mathbb{R}}$ can be mapped to any other triple by an element of Möb(H)

$\Rightarrow$ all such triangles are congruent.

Pf of Prp 2.8.5



Let A vertex w/ angle α

B β

C γ

W.l.o.g. $\overleftrightarrow{AC} = i\mathbb{R}_{>0}$, $Re(B) > 0$

Assume finite triangle, let

m = center of circle containing [A, B]

r = euclidean radius —— " ——

Let $y_0 > |C|$, $P = iy_0$, $Q = \mathrm{Re}B + iy_0$,

$ABQP$ = region bdd by $[A,B]$, $[B,Q]$, QP & $[P,A]$

where QP is the horizontal curve between Q & P.

<u>Claim</u> $A_{\mathbb{H}}(ABQP) = \pi - u - v - \dfrac{m + r\cos v}{y_0}$, u,v as in picture

Integrate vertical slices 

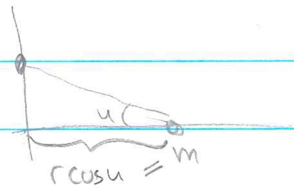
<u>Pf</u> $x = m - r\cos t$, $t \in [u, \pi - v]$ $\Rightarrow$ $\dfrac{dx}{dt} = r\sin t$

$y \in [r\sin t, y_0]$

$$\Rightarrow A_{\mathbb{H}}(ABQP) = \int_{t=u}^{\pi-v} \left(\int_{r\sin t}^{y_0} \frac{dy}{y^2} \right) \underbrace{r\sin t\,dt}_{dx} =$$

$$\int_{t=u}^{\pi-v} \left[-\frac{1}{y} \right]_{r\sin t}^{y_0} r\sin t\,dt = \int_{t=u}^{\pi-v} \left[\frac{1}{r\sin t} - \frac{1}{y_0} \right] r\sin t\,dt$$

$$= \pi - v - u + \frac{1}{y_0} \left[-r\cos t \right]_u^{\pi-v}$$

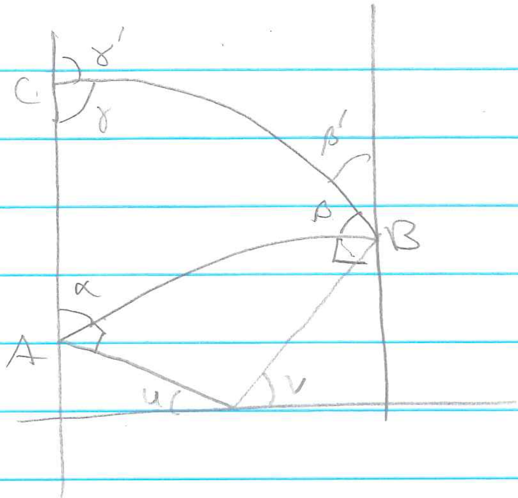


$$= \pi - v - u + \frac{1}{y_0} \left[\underbrace{r\cos u}_{m} + r\cos v \right]$$ △ Claim

Since this holds for all $y_0 > C$ $\Rightarrow$

area of the asymptotic triangle $AB\tilde{C}$ w/ $\tilde{C} = \infty$

is given by $A_{\mathbb{H}}(AB\tilde{C}) = \pi - u - v - 0$



<u>Note</u> $u=\alpha$, $v=\beta+\beta'$

and $\beta'\to 0$ as $|C|\to\infty$

$\Rightarrow A_{\mathbb{H}}(AB\tilde{C}) = \pi-\alpha-\beta = \pi-\alpha-\beta-\gamma$ since $\gamma=0$

!!! Finite area.

Next if $A\to 0$ or $B\to\widehat{\mathbb{R}}$ we still have

$$A_{\mathbb{H}}(ABC) = \pi-u-v = \begin{cases} \pi-u=\pi-\alpha & \text{if } B\to\widehat{\mathbb{R}} \\ \pi & \text{if } A,B\to\widehat{\mathbb{R}} \end{cases}$$

$\Rightarrow$ formula holds also for doubly or triply asymptotic triangles.

<u>Finite</u> $A_{\mathbb{H}}(ABC) = A_{\mathbb{H}}(AB\infty) - A_{\mathbb{H}}(CB\infty)$

$$= (\pi-\alpha-\beta-\beta') - (\pi-\gamma'-\beta')$$

$$= \pi-\alpha-\beta-\beta' \underbrace{-\pi+\gamma'}_{-\gamma} + \beta' = \pi-\alpha-\beta-\gamma \qquad \square$$

$\gamma+\gamma'=\pi$