Part II of the course

Surfaces & Riemannian geometry

* Topological classification of surfaces (next 3 lectures)

* "Homogeneous" geom. str on surfaces
  - Euclidean
  - Hyperbolic
  - Elliptic

* Non-homogeneous geom str on surfaces (Riemannian geometry)

looks euclidean, on some part, hyperbolic somewhere else & also parts that looks elliptic. Measured by the curvature §5.5, and lines = geodesics §5.6.

Today * Define what we mean by a surface

* Introduce tools from differential geometry to study them

Def A surface is a

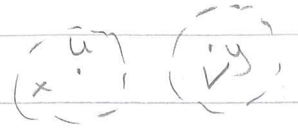
2-dimensional topological manifold

= a Hausdorff topological space s.t. every point has a nbhd which is homeomorphic to $\mathbb{R}^2$

"locally homeomorphic to $\mathbb{R}^2$"

Recall $\ast$ X Hausdorff: For any 2 pts $x \neq y \in X$

$\exists$ nbhds $U \ni x$, $V \ni y$ s.t. $U \cap V = \emptyset$



$\ast f : X \to Y$ homeomorphism "continous bijection

w/ continous inverse.

Ex of surfaces

- $\mathbb{R}^2$, S^2 (sphere), $T^2 = S^1 \times S^1$ (torus), $P^2 = S^2/_{x \sim -x}$ (real projective plane)

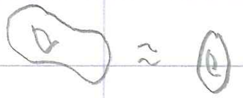
- Open subsets of other surfaces, e.g. $\mathbb{H} \supset \mathbb{D} \subset \mathbb{R}^2$
- Complex curves = zero set of cplx analytic fcns $f(z,w) = \mathbb{C}^2 \to \mathbb{C}$ where $\frac{\partial f}{\partial z} \neq 0$ or $\frac{\partial f}{\partial w} \neq 0$

  Has cplx dim 1 = real dim 2.

Aim Classify all surfaces up to homeomorphism

"topological classification"

<u>Recall</u> X, Y "same" topological space if

 $\exists$ homeo $f: X \to Y$. Then write $X \approx Y$.

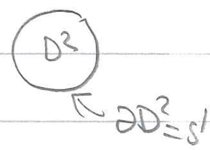
<u>Note</u> Enough to classify connected surfaces
(since a given surface is a disjoint union of its connected comp.)

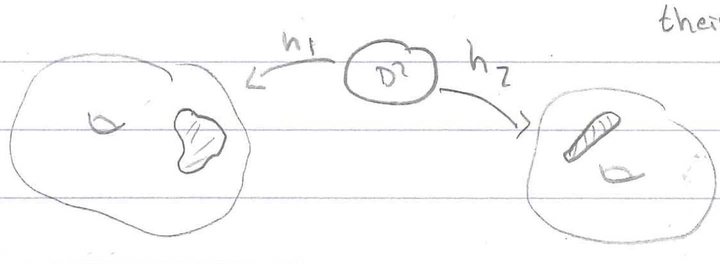
For this classification, we will use the

Connected sum

Given 2 surfaces M_1, M_2, let

$h_i : D^2 \to M_i$, $i=1,2$, be $\underbrace{\text{embeddings}}_{\text{homeos onto their image}}$ of the closed 2-disk



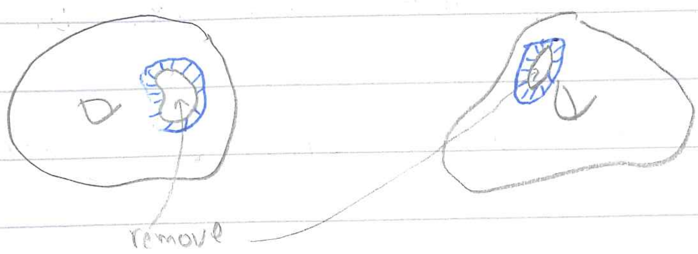


The <u>connected sum</u> $M_1 \# M_2$ of M_1 & M_2 is

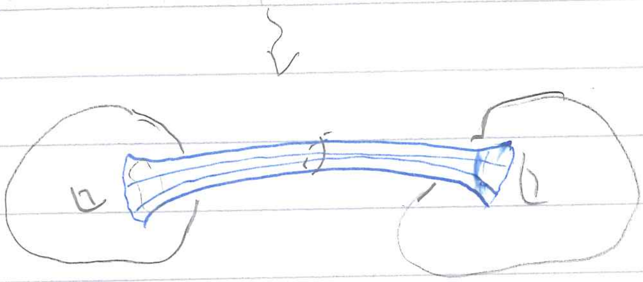
the surface

$$M_1 \# M_2 = (M_1 - h_1(\text{int}\, D^2)) \cup_f (M_2 - h_2(\text{int}\, D^2))$$

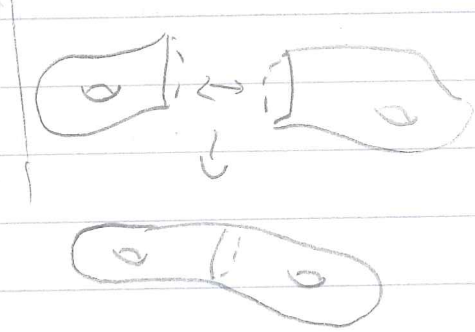
where $f = h_2 \circ h_1^{-1}|_{h_1(S^1)} : h_1(S^1) \to h_2(S^1)$



blue area
$= S^1 \times [0, \varepsilon]$.
Use this to glue



Homeomorphic picture



<u>Rmk</u> Does not depend on choice of h_1, h_2 in dim 2.

<u>Lma</u> i) $M \# S^2 \approx M$

ii) $M_1 \# M_2 \approx M_2 \# M_1$

iii) $M_1 \# (M_2 \# M_3) \approx (M_1 \# M_2) \# M_3$

$\Rightarrow$ # makes {surfaces} into monoid

<u>Q</u> Draw pictures of i) – iii) to convince yourself this is true

<u>Def</u> $S(m,n) = \underbrace{T^2 \# T^2 \# \ldots \# T^2}_{m} \# \underbrace{P^2 \# \ldots \# P^2}_{n}$

$S(0,0) = S^2$

Our goal is to prove the following:

Thm 3.1.14 Any $\underbrace{\text{closed}}^{\text{cpct w/o bdry}}$, connected surface M is homeomorphic to a surface of type $S(m,n)$ where $n=0,1$ or 2. Moreover, m and n are then uniquely determined by M.

To prove this, we will use the theory of combinatorial surfaces (§3, next week) + differential geometry

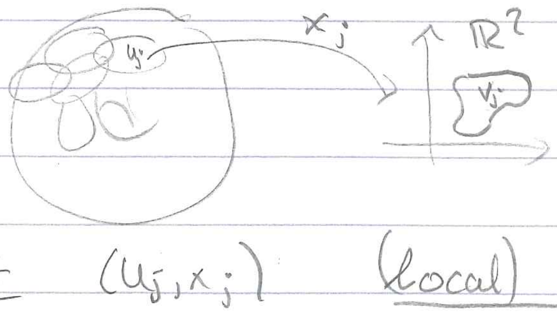
→ to view our surfaces as

Differentiable manifolds (§4.1)

Surface M locally homeo to $\mathbb{R}^2$ $\Rightarrow$

M has an open covering $\{U_j\}_{j\in J}$ + homeos

$x_j : U_j \longrightarrow x_j(U_j) \subset \mathbb{R}^2$ where $x_j(U_j) = V_j$ open in $\mathbb{R}^2$



Def (U_j, x_j) (local) chart on M

(V_j, x_j^{-1}) (local) parametrization / coordinate patch

$\{(U_j, x_j)\}_{j\in J}$ atlas

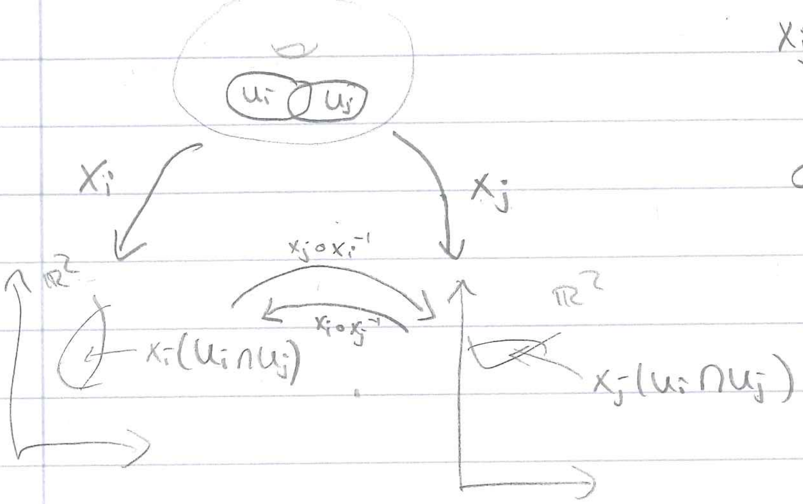
$\{(V_j, x_j^{-1})\}_{j\in J}$ —‖—

$\{(U_j, x_j)\}_{j\in J}$ <u>differentiable atlas</u> for M if

for all charts U_i, U_j s.t. $U_i \cap U_j \neq \emptyset$, the

<u>coordinate transformations</u> $x_i \circ x_j^{-1} : x_j(U_i \cap U_j) \to x_i(U_i \cap U_j)$

$x_j \circ x_i^{-1} : x_i(U_i \cap U_j) \to x_j(U_i \cap U_j)$



are differentiable

$\rightsquigarrow$ <u>differentiable structure</u> on M

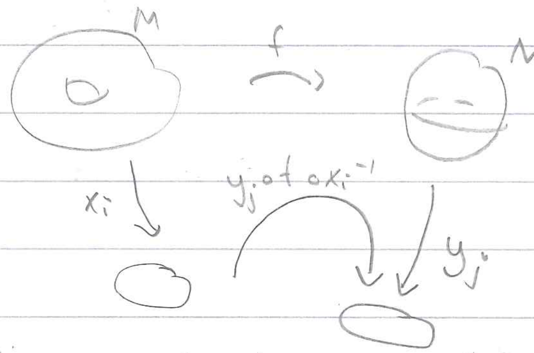
<u>Def</u> A surface (manifold) w/ a choice of differentiable structure is called a <u>differentiable surface</u> (mfd)

<u>Rmk</u> * Will assume $x_i \circ x_j^{-1} \in C^\infty$ $\rightsquigarrow$ smooth surfaces

* 2 diffable structures are equivalent $\iff$ their defining atlases $\{(U_i, x_i)\}$ & $\{(V_j, y_j)\}$ satisfies that all compositions $y_j \circ x_i^{-1}$ & $x_i y_j^{-1}$ are diffable (when defined)

$\iff \{(U_i, x_i)\} \cup \{(V_j, y_j)\}$ diffable atlas

<u>Def</u> If M, N smooth surfaces w/ atlases $\{(U_i, x_i)\}$ and $\{(V_j, y_j)\}$, resp. Then a map $f: M \to N$ is <u>differentiable / smooth</u> if all maps $y_j \circ f \circ x_i^{-1}$ are (whenever they are defined)



<u>Rmk</u> If $N = \mathbb{R}^n$, take trivial atlas $\rightsquigarrow$ $f \circ x_i^{-1}$ should be diffable

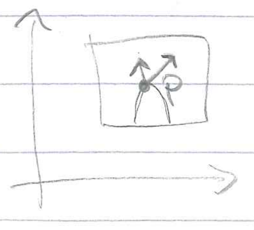
<u>Def</u> $f: M \to N$ is a <u>diffeomorphism</u> if smooth homeo,
(iso in category of smooth mfds)

and M & N are then <u>diffeomorphic</u>.

Derivatives

Must first define <u>tangent planes</u> of surfaces (§5.1)

In $\mathbb{R}^2$



Tangent plane at $p = T_p\mathbb{R}^2$

$= \{$ tangent vectors of curves through $p \}$

$= \{ \gamma'(0) \mid \gamma \subset \mathbb{R}^2$ curve w/ $\gamma(0) = p \}$

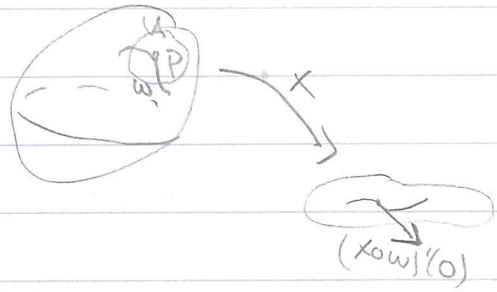
$\simeq$ lin space $\mathbb{R}^2$ by translating p to 0.

$\Rightarrow$ tangent vectors to S at p should be linear approximations of curves on S:

Let $p \in S$, $\Omega_p(S) = \{ \omega : J \to S;\ J$ open interval containing 0, $\omega(0) = p \}$

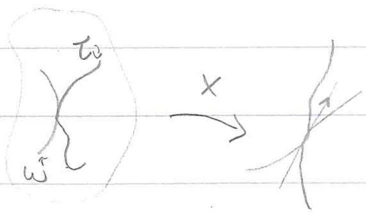
Let $x : U \to \mathbb{R}^2$ local chart around p

$\Rightarrow x \circ \omega$ curve at $x(p) \in \mathbb{R}^2 \Rightarrow (x \circ \omega)'(0) \in T_p\mathbb{R}^2$



Define equivalence rel $\sim$ on $\Omega_p(S)$ by

$\omega \sim \tau \Leftrightarrow (x \circ \omega)'(0) = (x \circ \tau)'(0)$



Let $T_pS = \Omega_p(S)/\sim$. This is well-def, and in fact a vsp, the <u>tangent plane of S at p</u>:

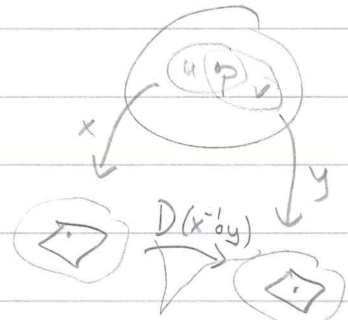
<u>Lma 5.1.1</u> (1) $\sim$ does not depend on the choice of local chart (x, U)

(2) $\overline{\omega} \mapsto (x \circ \omega)'(0)$ induces a bijection $T_pS \approx \mathbb{R}^2$

equivalence class

(3) The bijection in (2) defines a vsp str on T_pS which is indep. of chart x.

Pf (1), (3) follows from



matrix of partial derivatives

Jacobian of $x^{-1} \circ y$ is an isomorphism $T_{x(p)}\mathbb{R}^2 \simeq T_{y(p)}\mathbb{R}^2$? since $x^{-1} \circ y$ diffeo.

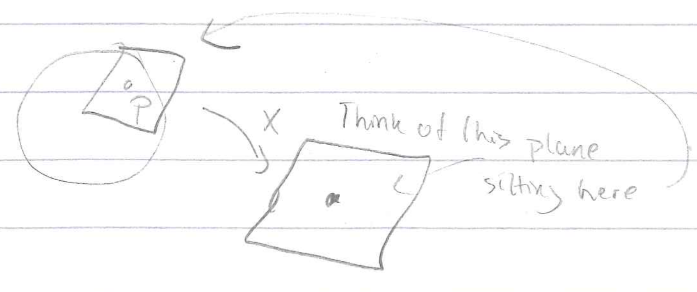
(2) Injective by def

Surj: If $v \in \mathbb{R}^2$, let $\boxed{\omega(t) = x^{-1}(x(p) + tv)}$

def for t so small so that in $x(U)$

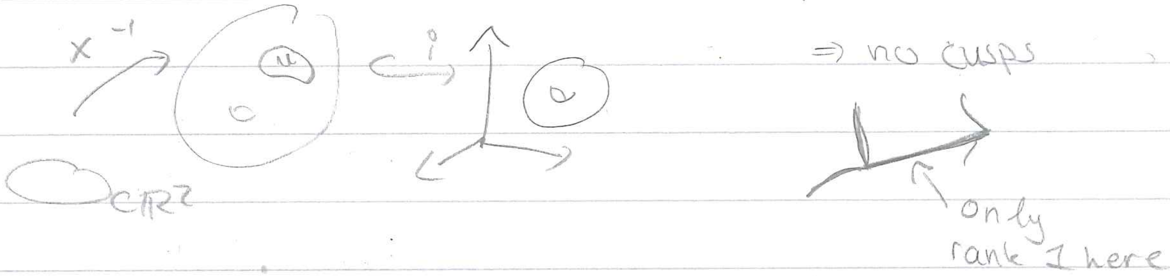
Clearly $(x\omega)'(0) = v$. [and this is the iso] : □

$\rightsquigarrow$



Ok, since for surfaces in $\mathbb{R}^3$, this is what we get.

Def 5.1.3 Let $S \subset \mathbb{R}^3$ be a smooth surface, let $\iota : S \hookrightarrow \mathbb{R}^3$ be the inclusion map. Then S is a regular surface if for every local chart x of S the Jacobian of $\iota \circ x^{-1}$ has rank 2 at every pt



Notation From now on $S \subset \mathbb{R}^3 \Rightarrow S$ regular

Rmk Can show S regular $\Leftrightarrow$ locally on the form $\{(x,y,z) \in \mathbb{R}^3 : F(x,y,z) = 0\}$ for some smooth $F : \mathbb{R}^3 \to \mathbb{R}$ w/ $\nabla F \neq 0$ along zero-set.

Ex $S^2 = \{\underbrace{x^2 + y^2 + z^2 - 1}_{F} = 0\} \Rightarrow \nabla F = (2x, 2y, 2z)$

$$T^2 = \{(\sqrt{x^2+y^2} - 2)^2 + z^2 - 1 = 0\}$$

Q Draw this

<u>Ex</u> Graphs of smooth fcns $f(u,v)$ defined on open subsets $\mathcal{O} \subset \mathbb{R}^2$ param. by $\underline{x}(u,v) = (u, v, f(u,v))$

param, not chart

<u>Ex</u> S^2 w/ $\mathcal{O} = D^2 \subset \mathbb{R}^2$ open unit disk, $f = \pm\sqrt{1-u^2-v^2}$

+ rotation  ← not equator

rotate →  + more rotations 

Tangent planes

If $S \subset \mathbb{R}^3$ then $T_pS \simeq$ plane in $\mathbb{R}^3$ containing p & all tangent lines to curves on S through p

<u>Q</u> Why?

$\simeq$ lin subspace of $\mathbb{R}^3$ via translation taking p to 0

<u>Rmk</u> If $S = \{F(x,y,z) = 0\}$ then T_pS is determined by the normal $\nabla F(x,y,z)$ at p.

Inverse function thm $\Rightarrow$

<u>Prp 5.1.9</u> A regular surface $S \subset \mathbb{R}^3$ coincides w/ a <u>graph</u> of a smooth fcn in a nbhd of every pt.

possibly after rotation / translate