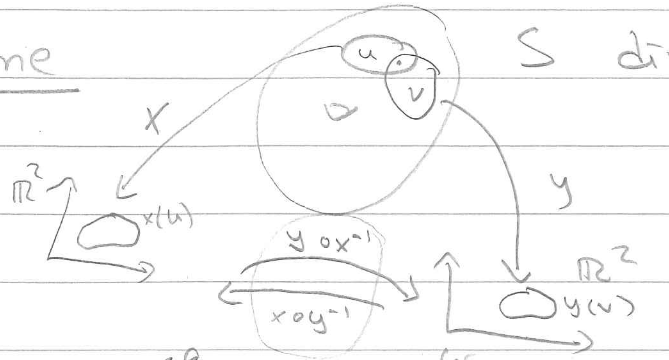
Last time S diffable/smooth if

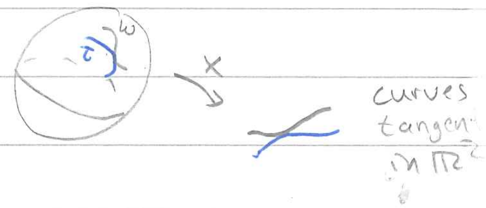


diffable (for $\forall$ coord transformations)

T_pS = equivalence classes of curves $\omega : \underset{\substack{\frown\\ t}}{J} \to S$ s.t. $\omega(0) = p$,

$\tau \sim \omega$ if

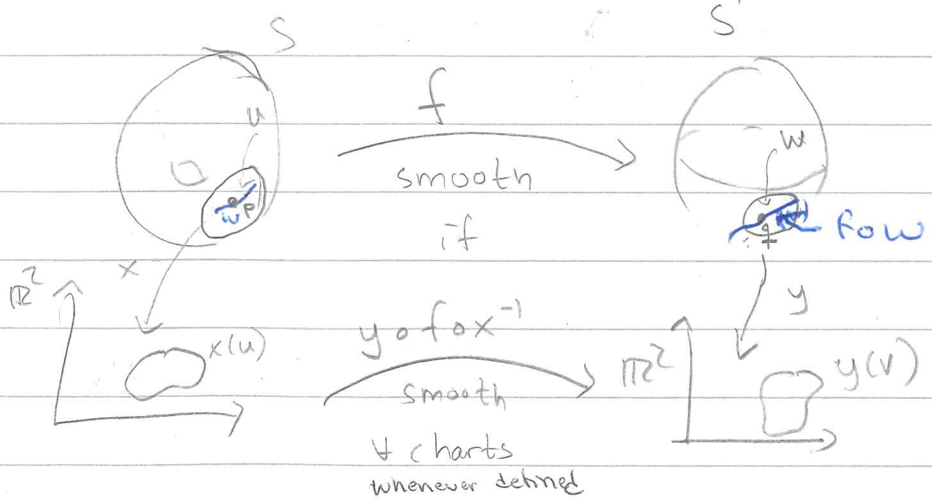
$(x \circ \omega)'(0) = (x \circ \tau)'(0)$



Notation $\omega'(0)$ = equivalence class of ω

= the tangent vector of ω at p.

Also



$\Rightarrow$ if $p \in S$, $q = f(p)$, $\omega \in \Omega_p(S) \Rightarrow f \circ \omega \in \Omega_q(S')$

$\rightsquigarrow$

Derivatives of smooth maps (§5.1)

Def 5.1.6 The derivative of f at p is the map

$df_p : T_pS \to T_q S'$ defined by $df_p(w'(0)) = (f\circ w)'(0)$

Well-defined:

Lma 5.1.5 If $w'(0)=\tau'(0)$ then $(f\circ w)'(0)=(f\circ\tau)'(0)$.

Pf $(yfw)'(0) = ((yf\underbrace{x^{-1})(x}_{=id}\circ w))'(0) \underset{\text{chain rule}}{=} \underbrace{J}_{\text{Jacobian}}(yfx^{-1})_{x(p)}(x\circ w)'(0)$

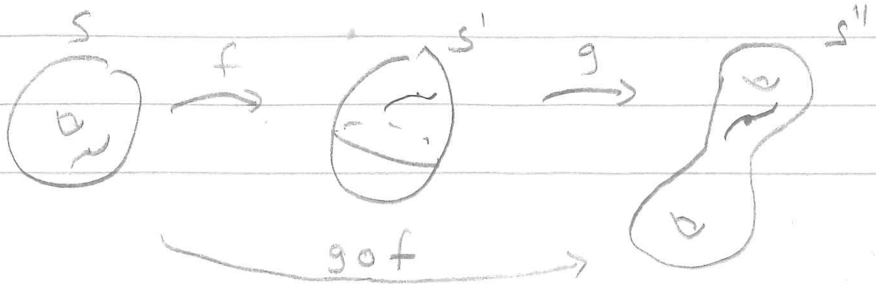
$\underset{\text{by assumption}}{=} J(y\circ f\circ x^{-1})_{x(p)}(x\circ\tau)'(0) = (y\circ f\circ\tau)'(0)$ $\square$

More generally:

Lma 5.1.8 Let $f: S\to S'$ smooth. Then

(1) df_p is a linear transformation $\forall p\in S$

(2) (Chain rule) If $g: S'\to S''$ another smooth map, then $d(g\circ f)_p = dg_q \circ df_p$.



Pf (2) : $d(gf)_p(w'(0)) = (g \circ f \circ w)'(0) = dg_{f(p)}((f \circ w)'(0))$

$= dg_{f(p)}(df_p(w)'(0))$.

(1) Note If $x : U \to \mathbb{R}^2$ chart w/ $p \in U$ then

$dx_p : T_p S \to T_{x(p)} \mathbb{R}^2$, $dx^{-1}_{x(p)} : T_{x(p)} \mathbb{R}^2 \to T_p S$

→ are the bijections from Lma 5.1.1 $(v \in \mathbb{R}^2 \mapsto x^{-1}(x(p) + tv))$
$[dx_p(w'(0)) = (x \circ w)'(0),\ dx_p^{-1}(v) = (x^{-1} \circ (x(p) + tv))'(0)]$

$\Rightarrow df_p = d(y^{-1} \circ (y \circ f \circ x^{-1}) \circ x)_p = dy^{-1}_{y(f(p))} \circ \underbrace{d(y \circ f \circ x^{-1})_{x(p)}}_{\substack{\text{lin transformation} \\ \mathbb{R}^2 \to \mathbb{R}^2}} \circ dx_p$

$\Rightarrow df_p$ is the composition of 3 linear transformations. ▽

Q

§5.2 Orientation

Will define orientability & non-orientability for surfaces, and see that this is an invariant under homeomorphisms.

(If $M \cong N$ then M orientable iff N is)

<u>Recall</u> An orientation of a vsp is an equivalence class of ordered bases,

2 bases are equivalent $\Leftrightarrow$ the transition matrix between them has pos. determinant

($\Rightarrow$ get 2 equivalence classes)

<u>Ex</u> in $\mathbb{R}^2$ $\quad (\overset{(1,0)}{e_1}, \overset{(0,1)}{e_2}) \sim (-e_1, -e_2) \sim (-e_2, e_1) \not\sim (e_1, -e_2)$

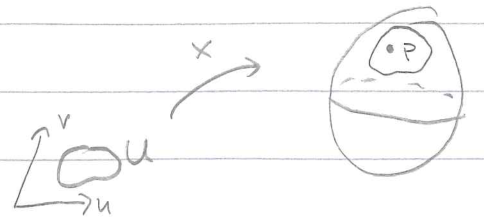
Surfaces

<u>Def 5.2.1</u> The surface S is <u>orientable</u> if it admits a diffable atlas s.t. the Jacobians of all the coord. transformations have positive determinant everywhere. ($|J(x_j \circ x_i^{-1})| > 0$)

An <u>orientation</u> of S is a choice of such a maximal atlas

<u>Rmk</u> $p \in S \Rightarrow T_pS$ has 2 possible orientations.

A <u>parametrization</u> (inverse of chart) $x : \underset{\mathbb{R}^2}{U} \longrightarrow S$ gives a natural (oriented) basis for T_pS $\forall\, p \in x(U)$:



Choose orient on $\mathbb{R}^2$ given by $((1,0),(0,1))$.

This induces an orientation on T_pS by $(dx(1,0), dx(0,1))$, the <u>positive orientation</u> if (U,x) belongs to the chosen atlas in Def 5.2.1

<u>Notation</u> If (u,v) coord on $\mathbb{R}^2$, then $x = x(u,v)$ & we write

$x_u = dx(1,0) \in T_pS$, $x_v = dx(0,1) \in T_pS$ "partial derivatives"

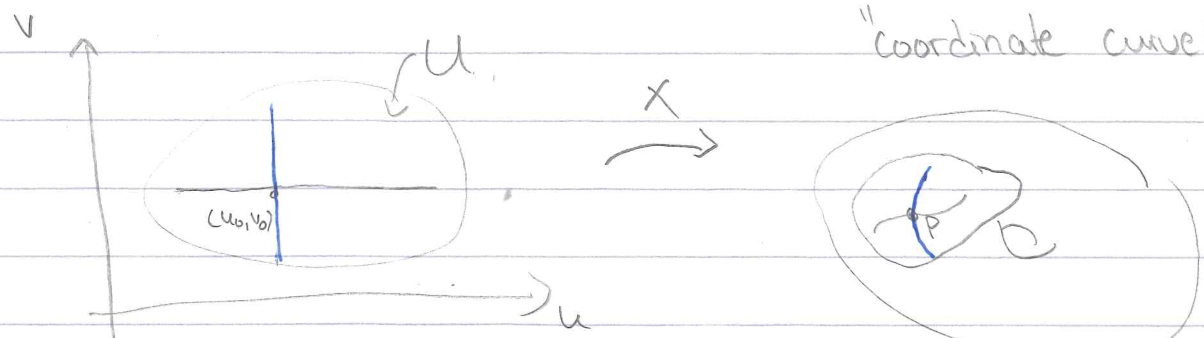
⚠ Only equivalence classes of curves.

In fact, if $p = x(u_0, v_0)$ then

$x_u = x_u(u_0, v_0)$ is repr by the curve $t \mapsto x(u_0 + t, v_0)$

$x_v = x_v(u_0, v_0)$ —— || —— $t \mapsto x(u_0, t + v_0)$

"coordinate curves"



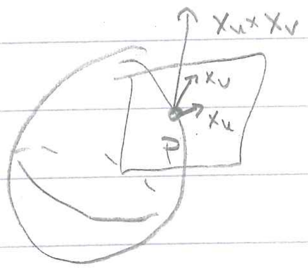
⚠ No std notation, sometimes $x_u = \frac{\partial}{\partial u}$, $x_v = \frac{\partial}{\partial v}$, sometimes other variants. Always check the text you read for which convention is used.

Rmk If $S \subset \mathbb{R}^3$ then $x = \iota \circ x : \overset{\subset \mathbb{R}^2}{U} \to \mathbb{R}^3$ ⚠

x_u, x_v are the true partial derivatives of x,

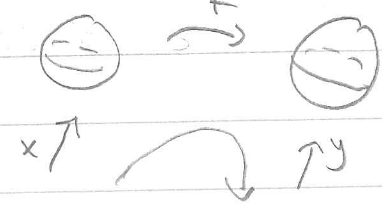
$J(x) = \begin{bmatrix} | & | \\ x_u & x_v \\ | & | \end{bmatrix}$. S regular $\Leftrightarrow$ $x_u, x_v \in \mathbb{R}^3$ lin indep.

$\Leftrightarrow$ $x_u \times x_v \neq 0$ everywhere, and this gives a normal vector to the tangent plane



Def If $f : S \to S'$ is a (local) diffeo between oriented surfaces, then f is orientation preserving if df_p maps positively oriented bases to positively oriented bases $\forall p$.

not nec 1-1



$\Leftrightarrow$ $\mathrm{Det}(J(y^{-1} \circ f \circ x)) > 0$ $\forall x, y$

Q Show that if S, S' diffeomorphic, then both either orientable or non-orientable.

[Given atlas $\{(U_j, x_j)\}$ for S defining an orientation, then $\{(U_j, f\circ x_j)\}$ gives an atlas for S' w. $\det((f\circ x_j)\circ(f\circ x_i)^{-1}) = \det(x_j\circ x_i^{-1}) > 0$.
Similarly use f^{-1} to define orientable atlas on S if S' orientable]

Characterization of orientability

For regular surfaces in $\mathbb{R}^3$:

Prp 5.2.2 If $S \subset \mathbb{R}^3$ regular surface, then:

S orientable $\Leftrightarrow$ it has a smooth normal vector field.

Pf "$\Rightarrow$" $N(p) = \dfrac{x_u \times x_v}{\|x_u \times x_v\|}$ & this is well-defined (does not depend on param. x from orientable atlas)

"$\Leftarrow$" Choose orient of $\mathbb{R}^3$. For each $p \in S$ choose orient on T_pS given by oriented tangent vectors $(v_1(p), v_2(p))$ s.t. $(v_1(p), v_2(p), N)$ repr. orient. of $\mathbb{R}^3$.

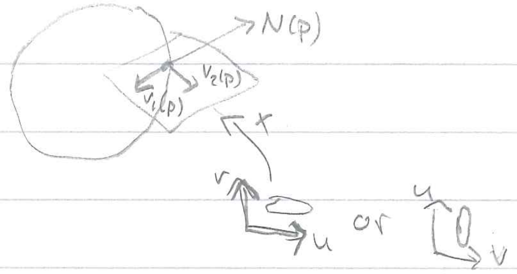
If $x : U \to S$, $x = x(u,v)$ param. of S at p s.t. $(x_u, x_v) \not\sim (v_1, v_2)$, choose instead $x = x(v,u)$ as param

$\Rightarrow dx : \mathbb{R}^2 \to T_pS$ orientation-preserving.

Do this for any param in the given atlas $\Rightarrow$

$J(x^{-1} \circ y) = dx^{-1} \circ dy$ orient-preserving for any

coord transformation in this atlas. $\square$



Prp T^2 is orientable.

Pf ⓐ Check that $T^2 \subset \mathbb{R}^3$ can be given by

$T^2 = \{F(x,y,z) = 0\}$ where

$F(x,y,z) = \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 - 1$, and

$\nabla F(x,y,z) = \left(2x\frac{(r-2)}{r}, 2y\frac{(r-2)}{2}, 2z\right)$ gives a normal

vector to T^2 at $T_{(x,y,z)}T^2$ $\left(r = \sqrt{x^2+y^2}\right)$

Since on T^2 we have $r \in [1,3]$ and $z = 0 \Rightarrow r = 1$ or $r = 3$

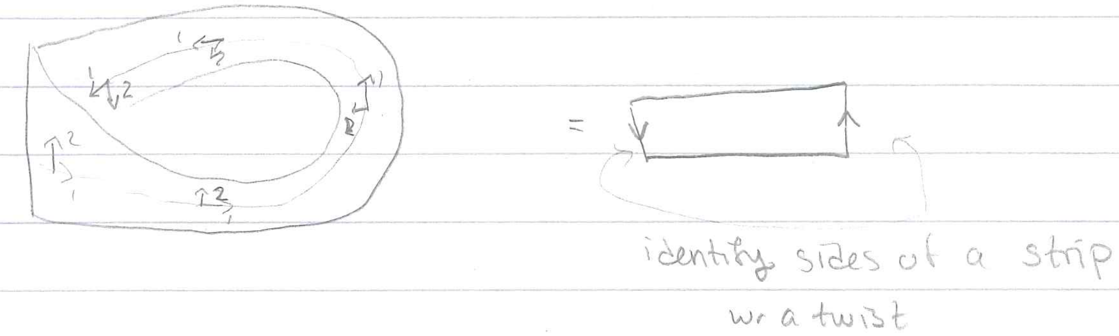
it follows that ∇F is smooth & $\neq 0$ on T^2

$\Rightarrow T^2$ has a smooth normal vector field, hence T^2 is orientable. $\square$

A more geometric / <u>intrinsic</u> characterization of orientability:

does not depend on embedding into $\mathbb{R}^3$

A <u>Möbius band</u> is the following surface:



<u>Rmk</u> Has 1 bdry component $\cong S^1$

$\boxed{\text{This is not orientable}}$

It has closed curves that are <u>not</u> orientation preserving.

<u>Def</u> A closed, regular curve $\alpha:[0,1]\to S$ is
$\alpha(0)=\alpha(1)$ $\quad$ $\alpha'(s)\neq 0$

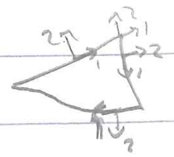
<u>orientation-preserving</u> if we can make a continous choice of orientatios of $T_{\alpha(s)}S$ $\forall s\in[0,1]$

(= a cont choice of basis $(\alpha'(s), v(s))$ of $T_{\alpha(s)}S$ s.t.

$(\alpha'(0), v(0)) \sim (\alpha'(1), v(1))$ as oriented bases)

Can also be defined for piecewise regular curves

= curves that are regular except at a finite # of pts



<u>Prp 5.2.3</u> A surface S is orientable $\Longleftrightarrow$ every closed, piecewise regular curve is orientation preserving.

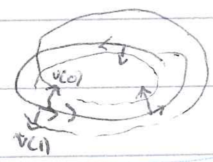
<u>Pf</u> See book

<u>Cor</u> (Lma 3.1.8) A surface is non-orientable $\Longleftrightarrow$ it contains a Möbius band.

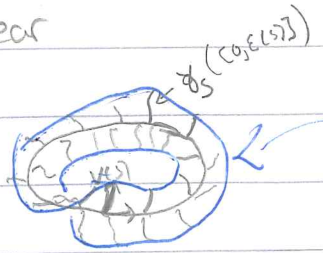
<u>Pf</u> Enough to prove S contains a Möbius band

$\Leftrightarrow$ contains a non-orientable curve

$\Rightarrow$ clear



Möbius band traced out by curves repr v.f along $\alpha(s)$ whatever was.

$\Leftarrow$ 

Let $\alpha(s)$, $s \in [0,1]$ be a non-orientable curve

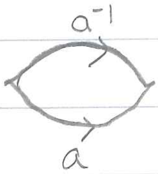
Cor $X \approx Y \Rightarrow (X \text{ orientable} \Leftrightarrow Y \text{ orientable})$

Pf X contains a Möbius band $\Leftrightarrow$ Y does □

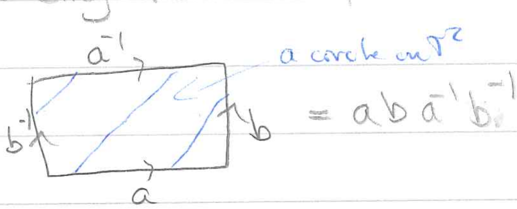
Lma P^2 contains a Möbius band.

Pf Use the following:

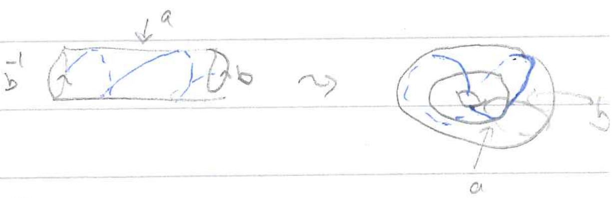
A surface can be given as a $2n$-gon where we identify sides pairwise according to some pattern:

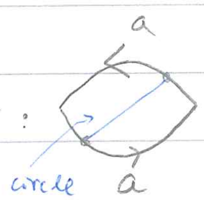
S^2: 

$= aa^{-1}$

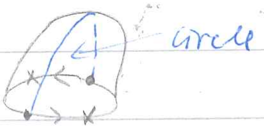
T^2: 

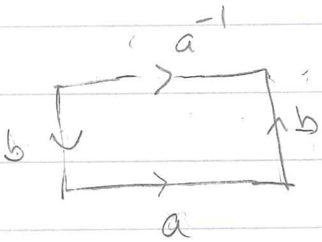
$= aba^{-1}b^{-1}$



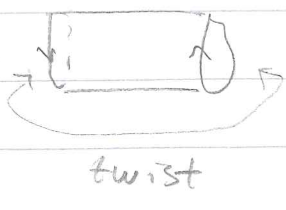
P^2: 

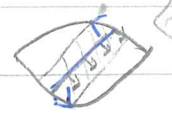
$= aa$



K^2 = Klein bottle 

$= aba^{-1}b$



From $P^2 =$  we get the Möbius band □

$\Rightarrow T^2 \not\approx P^2$.