<u>Recall</u> Want to

§3 Prove that if S surface then

$S \approx S(m,n) = \#_m T^2 \#_n P^2$ for some $n = 0,1,2$, and

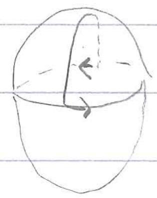
w/ m,n uniquely determined.

<u>Last time</u> Used orientability to show $T^2 \not\approx P^2$

$\ulcorner T^2$ has a regular emb. into $\mathbb{R}^3$ w/ a smooth

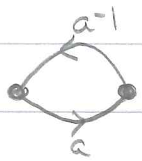
normal v.f. $\Rightarrow$ orientable.

P^2 contains a Möbius band $\Rightarrow$ non-orientable.



And orientability preserved

by homeos $\lrcorner$

Used the picture $P^2 =$ 

. In more detail:

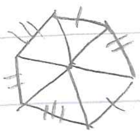
<u>Fact</u> Any closed surface can be <u>triangulated</u>

(Proven by Radó in 1925)

$\Rightarrow S \approx \bigcup$ triangles s.t. intersection of 2

triangles either empty, a common side or

a common vertex.

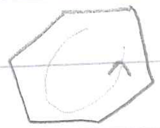


$\Rightarrow$ If S closed then $S \approx 2n$-gon

w/ edges pairwise identified

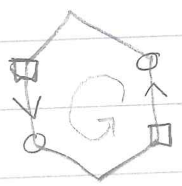
Describe this combinatorially:

Orient the boundary of the $2n$-gon.

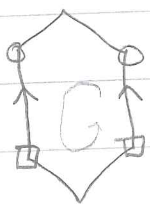


For any pairs of edges, 2 ways of identifying them:

Both orientation-preserving

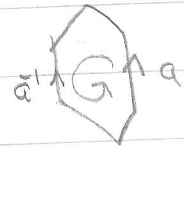


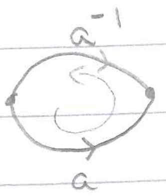
or orientation-reversing

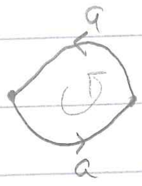


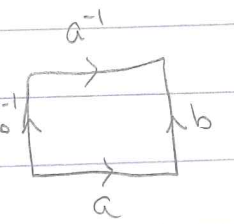
Label the edges w/ letters s.t. the edges which are identified get the same (a & a) if identified orient-preserving

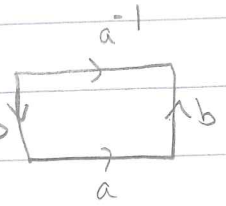
"inverse" (a & a^{-1}) if identified orient-reversing



Ex S^2: 

P^2: 

T^2: 

$K^2 =$ 

Notation Let $W = a \cdots a^{\pm 1} \cdots$ be the word starting at edge a & listing the edges counterclockwise along $\partial(2n\text{-gon})$.

Def A word W defining a surface is called admissible, and we write D^2/W for the corresponding surface.

Ex cont. $S^2 = D^2/a a^{-1}$, $P^2 = D^2/aa$, $T^2 = D^2/aba^{-1}b^{-1}$

$K^2 = D^2/aba^{-1}b$

Convention $D^2/\{\} = S^2$ where $\{\ \}$ is the empty word.

Also W^{-1} is the word W read clockwise,

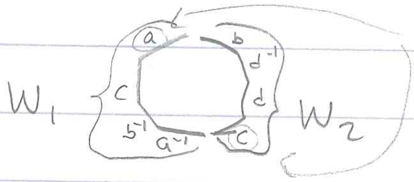
e.g. $(abc^{-1})^{-1} = cb^{-1}a^{-1}$

Lma 3.1.11 (i) If $W = W_1 W_2$ then

$$D^2/W_1W_2 \approx D^2/W_2W_1$$

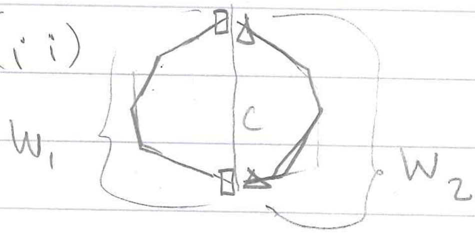
(ii) If W_1 & W_2 are admissible, then W_1W_2 is also admissible, and $D^2/W_1W_2 \approx D^2/W_1 \# D^2/W_2$

Pf (i) It doesn't matter where we start reading



can start at any pt.

(ii)

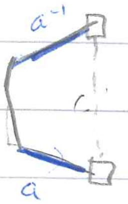
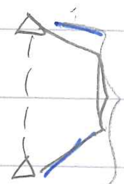


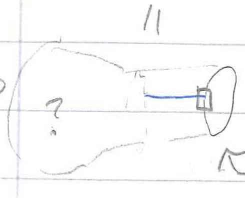
Cut disk along line c

W_1 admissible $\Rightarrow$ $\square$s identified in D^2/W_1

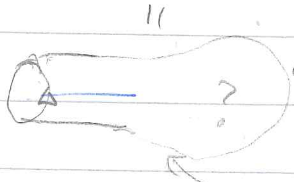
W_2 — " — $\Rightarrow$ Δ's identified in D^2/W_2

$\Rightarrow$ c gives a circle in both D^2/W_1 & D^2/W_2

$\Rightarrow$  $= D^2/W_1 \setminus (\text{disk bdd by } c)$,  $W_2 = D^2/W_2 \setminus \left(\text{disk bdd by } c\right)$



the other identifications from W_1



the other identifications from W_2

$\Rightarrow$ May construct D^2/W_1W_2 by gluing these

$\Rightarrow D^2/W_1W_2 \approx D^2/W_1 \# D^2/W_2$ $\square$

Cor (1) $D^2/Waa^{-1} \approx D^2/W \# S^2 \approx D^2/W$

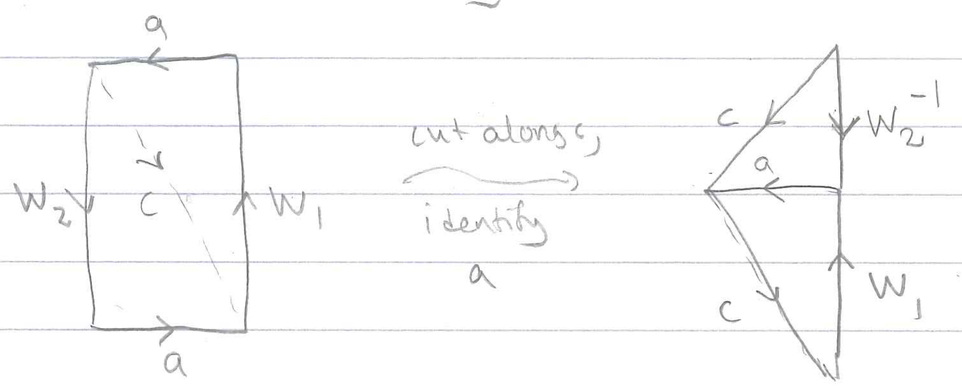
(2) $D^2/Waa \approx D^2/W \# P^2$

(3) $D^2/Waba^{-1}b^{-1} \approx D^2/W \# T^2$

(4) $D^2/W_1 a W_2 a \approx D^2/W_1 W_2^{-1} aa \approx D^2/W_1 W_2^{-1} \# P^2$

Pf (1) – (3) follows directly from Lma 3.1.11 (ii)

(4)



$\square$

Ex The Klein bottle $\approx P^2 \# P^2$

$$\begin{array}{ccc} \text{??} & & \wr\wr \\ D^2/\underbrace{b}_{W_1}a\underbrace{b^{-1}}_{W_2}a & \approx & D^2/bbaa \end{array}$$

Lma (Thm 3.1.10) Any closed, connected surface S is homeo to some $S(m,n)$, $n,m \in \mathbb{Z}_{\geq 0}$

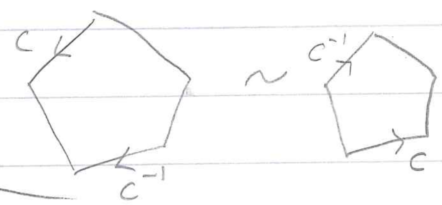
Pf Using (2) & (4): replace any duplicates kk w/ $\# P^2$

(1) $\Rightarrow$ can remove all strings kk^{-1}

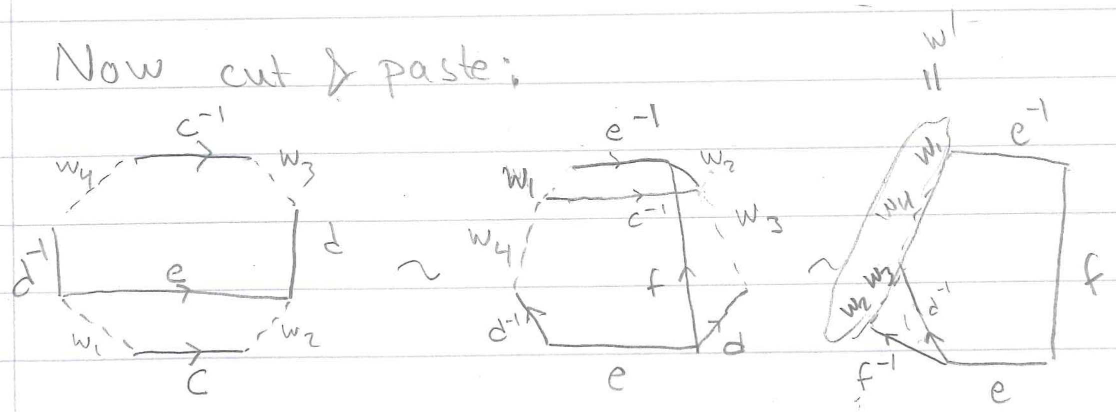
$\Rightarrow$ must either have $M \approx S^2 \underset{n}{\#} P^2 \approx S(0,n)$

or can find edges c & d occuring in the order $c \dots d \dots c^{-1} \dots d^{-1} \dots$

[possibly after switching labels c & c^{-1} and/or d & d^{-1}]



Now cut & paste:



Let $g = f^{-1}$ $\Rightarrow$ $D^2/W \approx D^2/W'geg^{-1}e^{-1} \approx D^2/W' \# T^2$,

where W' now is a shorter word than W

(at least 4 words shorter)

$\Rightarrow$ we may split off summands P^2 & T^2 until we get

$S^2 \# \underset{m}{\#} T^2 \# \underset{n}{\#} P^2 \approx S(m,n)$ $\square$

Cor S orientable $\Leftrightarrow$ $S \approx S(m,0)$.

Pf P^2 contains a Möbius band $\Rightarrow$ $S(m,n)$ contains a Möbius band if $n \neq 0$. $\square$

Last ingredient we need is:

Lma 3.1.13 $P^2 \# P^2 \# P^2 \approx T^2 \# P^2$

$\Rightarrow S(m,n) \approx S(m+1, n-2)$ if $n \geq 3$

Pf Recall $K^2 \approx P^2 \# P^2 \Rightarrow$

$$P^2 \# P^2 \# P^2 \approx K^2 \# P^2 \approx D^2 / \underbrace{ab}_{w_1}\underbrace{ab^{-1}}_{w_2}cc \overset{(4)}{\approx} D^2 / \underbrace{ab}_{w_1}\underbrace{acbc}_{w_2}$$

$$\approx D^2 / \underbrace{acbc}_{w_1}\underbrace{ab}_{w_2} \overset{(4)}{\approx} D^2 / aca^{-1}c^{-1}bb \approx T^2 \# P^2 \quad \square$$

Rmk/Q Can be proven in a more geometric way, f. ex. by drawing all disks above. Do that.

Now we are ready for:

Thm 3.1.14

Any closed, connected surface S is homeo to a surface
cpct w/o bdry
of the type $S(m,n)$, $n = 0, 1$ or 2. Moreover, m & n are then uniquely determined by S.

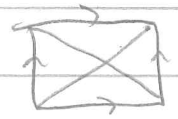
(*) [bottom] **Cor** S completely determined by $\chi(S)$ & whether S orientable or not

Pf Existence follows from Thm 3.1.10 & Lma 3.1.13.

Uniqueness Sketch Use the invariants
($X \simeq Y \Rightarrow$ have the same inv)
$\not\Leftarrow$ in general

orientability & the Euler characteristic $\chi(S)$, where

$\boxed{\chi(S) = v - e + s}$ if S triangulated by s triangles w/ e edges & v vertices.

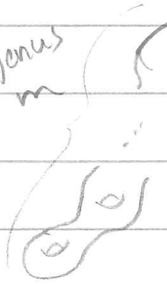
Ex  $s=4$, $e=6 \Rightarrow \chi(T^2)=4-6+2=0$, $v=2$

Fact $\chi(S)$ indep. on which triangulation we use

$\Rightarrow$ if $X \simeq Y$ then $\chi(X) = \chi(Y)$

Q: Why?

Exercise PSS3 $\chi(S(m,n)) = 2-2m-n$

genus m 

2 cases i) S orientable $\Rightarrow n=0 \Rightarrow \chi(S) = 2-2m$

$\Rightarrow m$ determined by $\chi(S)$ = the genus = # "holes"

ii) S non-orientable $\Rightarrow n=1$ or $n=2 \Rightarrow$

$\chi(S) = \underbrace{1-2m}_{\text{odd}}$ or $\chi(S) = \underbrace{-2m}_{\text{even}}$

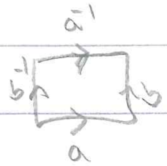
odd $\Rightarrow n=1$ and $m = \frac{1-\chi(S)}{2}$

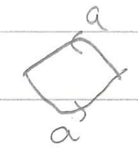
even $\Rightarrow n=2$ & $m = \frac{-\chi(S)}{2}$ $\square$

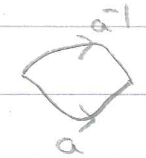
(*)

Last time Any closed, connected surface $S \approx D^2/W$

where D^2 a $2n$-gon, W word of $a, b, \dots, a^{\pm 1}, b^{\pm 1}, \dots$

$T^2 =$ 

$P^2 =$ 

$S^2 =$ 

and $S \approx S(m,n) = \#_m T^2 \#_n P^2$ for $n = 0,1,2$ and m & n then uniquely determined by the invariants orientability & $\chi(S)$ (Euler char)

Today § 4.2 Geometric structures on surfaces

Will study the question:

Given a surface S, when can we find an atlas for S s.t. the surface looks

* euclidean everywhere

or * hyperbolic —"—

or * elliptic (spherical) —"— ?

Clearly, on each coord patch we can put such a structure, but need to be able to glue them together ↝
This will be characterized in terms of coord transformations.

Let (X, G) be one of the pairs

- $(\mathbb{H}, \text{Möb}(\mathbb{H}))$ (or $(\mathbb{D}, \text{Möb}(\mathbb{D}))$),
- $(\mathbb{R}^2, E(2))$ where $E(2)$ is the gp of Euclidean isometries
- $(S^2, O(3))$, where $O(3)$ is the gp of spherical isometries (orthogonal matrices)

<u>Def</u> A surface has a geometric structure modeled on (X, G) (a hyperbolic / Euclidean / elliptic str, resp) if it has an atlas $\{(U_i, x_i)\}_{i \in I}$ s.t. all the coord transformations are restrictions of elements of G.

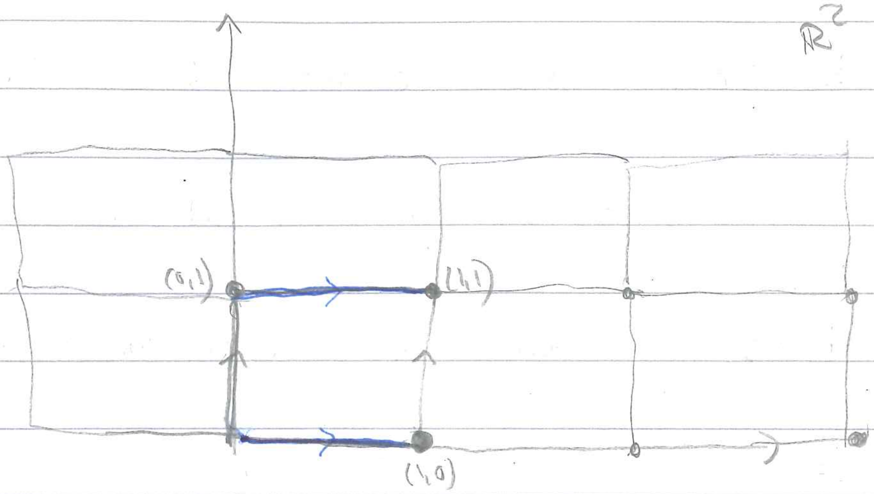
$(x_j \circ x_i^{-1} : U^{\subset \mathbb{R}^2} \to V^{\subset \mathbb{R}^2}$ in $\text{Möb}(\mathbb{H})$, $E(2)$ or $\underset{\text{whatever this means}}{\underline{O(3)}})$.

<u>Ex</u> (Euclidean) $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (w/ quotient topology)
$\left(\mathbb{Z} \subset \mathbb{R} \text{ subgp of additive gp } \mathbb{R}\right)$

$= \mathbb{R}^2/\Gamma$ where $\Gamma \subset E(2)$ subgp gen by

$\gamma(s,t) = (s+1, t)$, $\tau(s,t) = (s, t+1)$

$\mathbb{R}^2$



All squares are identified.

$\gamma(\rightarrow) = (\rightarrow)$, $\tau(\uparrow) = (\uparrow)$

Let $p : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{R}^2/\Gamma$ be the quotient map,

let $D(x,y)$ be the open disk of radius 1 centered

at $(x,y) \in \mathbb{R}^2 \Rightarrow P_{(x,y)} = p\big|_{D(x,y)} : D(x,y) \longrightarrow p(D(x,y))$ is

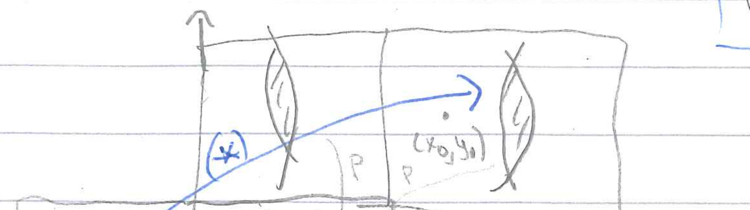
a homeo and $p(D(x_0,y_0)) = p(D(x_1,y_1))$

$\Leftrightarrow x_0 = m + x_1,\ y_0 = n + y_1$ for some $m, n \in \mathbb{Z}$

$\Rightarrow \{(P_{(x,y)}, D(x,y)\}_{(x,y)\in\mathbb{R}^2}$ gives an atlas for T^2

w/ coord transformations $\boxed{P^{-1}_{(x_1,y_1)} \circ P_{(x_0,y_0)}} = (x+m, y+n) \in E^2$

$*$



$P_{(x_0,y_0)}(D_{(x_0,y_0)}) \cap P_{(x_1,y_1)}(D_{(x_1,y_1)})$

⚠ Some problems defining "lines". If we take these to be the image of straight lines in $\mathbb{R}^2$, then

* lines w/ rational slopes $\mapsto$ closed curves
  $\Rightarrow$ betweenness a problem

* lines w/ irrational slopes $\mapsto$ dense subset in T^2
  $\Rightarrow$ 2 distinct "lines" may intersect ∞ many times

⚠ If $\Lambda \subset \mathbb{R}^2$ is another additive subgp $\Lambda \subset \mathbb{R}^2$ of rank 2, then $\mathbb{R}^2/\Lambda \approx T^2$ w/ a Euclidean str, but $\boxed{\text{not nec w/ a str that is equivalent}}$ to the one on $\mathbb{R}^2/\mathbb{Z}^2$.

union of the atlases do not nec give an euclidean atlas

<u>Ex 2</u> (Euclidean)

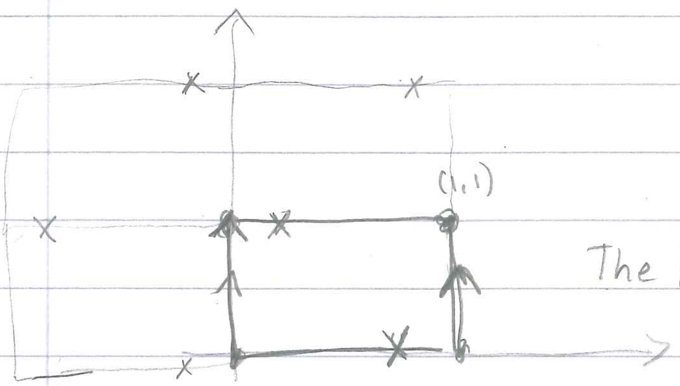
$X = \mathbb{R}^2$, $\Gamma \subset E(2)$ subgp gen by

$\gamma(s,t) = (s+1,t)$, $\tau(s,t) = (-s+1, t+1)$

$\Rightarrow$ the quotient map $p: \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ homeo when restricted to open disks of radius $= 1$

F(14) - 3

$\Rightarrow$ this gives an atlas for $\mathbb{R}^2/\Gamma$ w/ coord transformations in $\Gamma \subset E(2)$ $\Rightarrow$ we get an euclidean str

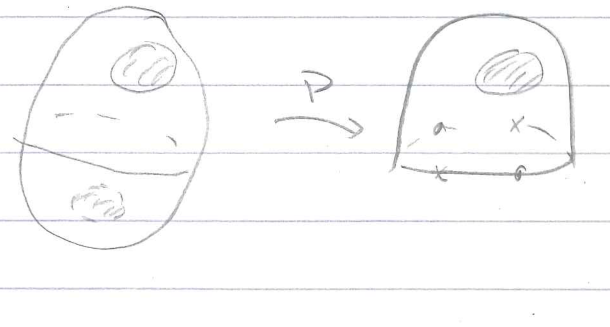


The Klein bottle!

Ex 3 Elliptic/spherical Only 2 ex, S^2 & P^2.

$$P^2 = S^2/\Gamma \quad w/ \quad \Gamma = \mathbb{Z}/2 = \{\pm 1\} \subset O(3)$$

acting on S^2 via the antipodal map



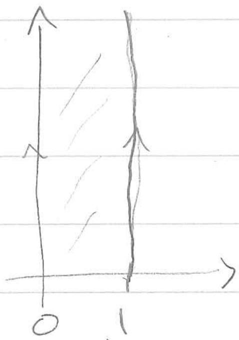
If $S^2 \subset \mathbb{R}^3$ then $p|_{D \cap S^2} : D \cap S^2 \to p(D \cap S^2)$ is a homeo if $D \subset \mathbb{R}^3$ is an open disk of radius 1. Use such disks w/ $D \cap S^2 \neq \emptyset$ to define an atlas for P^2 $\Rightarrow$ coord transformations are given by antipodal map $\subset \Gamma \subset O(3)$.

Hyperbolic:

Thm 4.2.1 Except for T^2, S^2, P^2 & K^2, every " surface" has at least one hyperbolic structure (In fact, ∞ many!)

Ex $f \in \text{Möb}^+(\mathbb{H})$ s.t. $f(z) = z+1 \Rightarrow \mathbb{H}/\langle f \rangle \approx S^1 \times \mathbb{R}$

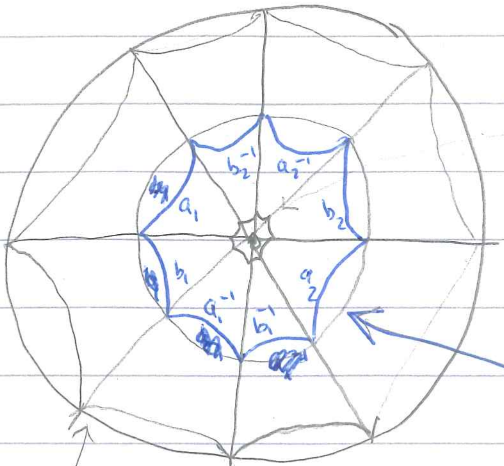
w/ hyperbolic structure



Ex $T^2 \# T^2 = \underbrace{D^2}_{\text{octagon}} / a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$

Need this to be a hyperbolic octagon w/ identifications of edges made via hyperbolic isometries.

Moreover, all vertices are identified to one pt & all angles are preserved $\Rightarrow \sum_{\substack{\text{bdry} \\ \text{vertices}}} \text{interior angles} = 2\pi$



small octagon has angles $\approx$ euclidean angles $= \frac{3\pi}{4}$

$\Rightarrow$ somewhere between must have (an octagon w/) angles $\frac{\pi}{4}$

$\rightsquigarrow$ our disk D^2

big octagon has angles 0

<u>Know</u> $\exists$ unique elements in $\gamma_i \in \text{Möb}^+(\mathbb{D})$

mapping a_i to a_i^{-1}, $i = 1,2$

$\tau_i \in \text{Möb}^+(\mathbb{D})$ mapping b_i to b_i^{-1} $i = 1,2$,

and doing this w/ the orientations prescribed in the figure.

Identify the edges of D^2 by these congruences

$\rightsquigarrow T^2 \# T^2$ w/ a hyperbolic str $\approx \mathbb{D}/_\Gamma$

where Γ generated by $\gamma_1, \gamma_2, \tau_1, \tau_2$.

<u>Q</u> Why doesn't this argument work to put a hyperbolic str. on T^2?