Next: Study "non-homogeneous" geometric structures on surfaces $\rightsquigarrow$ $\boxed{\text{Riemannian geometry}}$

§ 5.3 Riemannian surfaces

Non-homogeneous $\Rightarrow$ must study the local picture

Very local approximation $\approx$ tangent planes

Recall * In hyperbolic (also elliptic) geometry, things look euclidean when we zoom in

(e.g. small triangles $\approx$ euclidean triangles

small segments $\approx$ euclidean segments)

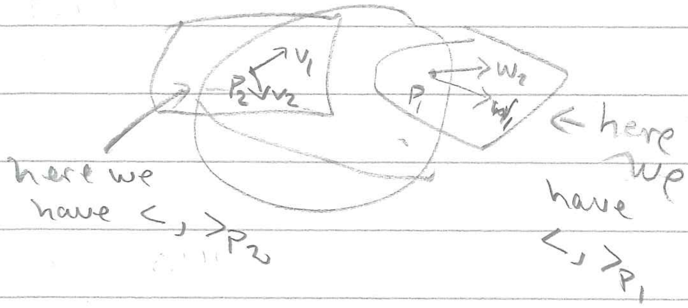
* Euclidean plane $= (\mathbb{R}^2, \langle , \rangle)$ ← inner product

$\langle , \rangle : \mathbb{R}^2 \to \mathbb{R}$ s.t.
$$\begin{aligned}
&\langle a x, y\rangle = a\langle x, y\rangle \quad a \in \mathbb{R},\ x, y \in \mathbb{R}^2 \\
&\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \quad x, y, z \in \mathbb{R}^2 \\
&\langle x, y\rangle = \langle y, x\rangle \\
&\langle x, x\rangle > 0 \text{ if } x \neq 0
\end{aligned}$$

$\Rightarrow$ equip all tangent planes on a surface w/ an inner product

Def 5-3.1 A Riemannian metric on S is a choice of inner product on every T_pS, varying smoothly w/ p.

Notation $\langle w_1, w_2 \rangle_p$ for the inner product of $w_1, w_2 \in T_pS$
sometimes skip



Smooth dependence If $x: U \xrightarrow{} S$, $U \subset \mathbb{R}^2$ local param, recall the "partial derivatives" $x_u = dx(1,0)$, $x_v = dx(0,1)$ which form a basis for $T_pS \Rightarrow$

$\langle , \rangle_p$ completely determined by

$$\boxed{E(p) = \langle x_u, x_u \rangle, \quad G(p) = \langle x_v, x_v \rangle, \quad F(p) = \langle x_u, x_v \rangle}$$

The inner product on S is smooth if E, F, G smooth $\forall$ local param.

Def A Riemannian surface is a surface w/ a $\underbrace{\text{Riemannian metric}}$

= Riemannian structure

Rmk can be generalized to mfds of any dimensions

⇝ Riemannian mfds

⚠ Difference between $\underbrace{\text{Riemann surface}}_{\text{cplx mfd of dim 1}}$ & above

Consequences/properties from having a Riemannian str.

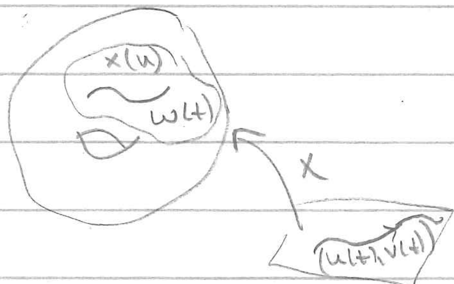
* Norm (of tangent vectors): $w \in T_pS \Rightarrow \boxed{\|w\|_p^2 = \langle w, w\rangle_p}$

$\Rightarrow$ if $w = \alpha x_u + \beta x_v$, then $\|w\|$

$$\|w\|_p^2 = \alpha^2 E(p) + 2\alpha\beta F(p) + \beta^2 G(p)$$

* Arc-length $w(t) = x(u(t), v(t))$ $t \in [a,b]$ C^1-curve in $x(U)$

$$\Rightarrow \boxed{s(t) = \int_0^t \|w'(t)\|_{w(t)}\, dt}$$



$w'(t) = u'(t)x_u + v'(t)x_v \Rightarrow$

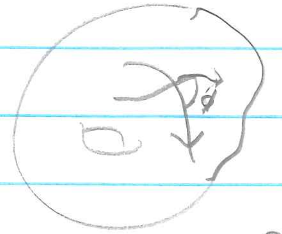
$$\|w'(t)\|_p^2 = E(p)(u'(t))^2 + 2F(p)u'(t)v'(t) + G(p)(v'(t))^2$$

$\Rightarrow$ line element $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$

* <u>Angle</u>: between curves $\alpha(t), \beta(t) \subset S$ s.t.

$\alpha(0) = \beta(0)$ given by

$\phi \in [0, \pi]$ s.t. $\langle \alpha'(0), \beta'(0) \rangle_{\alpha(0)} = \cos\phi \, \|\alpha'(0)\| \, \|\beta'(0)\|$



* <u>Area</u> $A(R)$ of region $R \subset X(U)$, $X: U \subset \mathbb{R}^2 \longrightarrow S$:

$$A(R) = \iint_{\Omega} \sqrt{EG - F^2} \, du \, dv \quad \text{where } \Omega \subset U \text{ s.t. } R = X(\Omega)$$

[<u>Motivation</u> If $R \subset \mathbb{R}^3$, then $A(R) = \iint_{\Omega} \left\| \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \right\| du\,dv$

$$= \iint_{\Omega} \|X_u \times X_v\| \, du \, dx$$

& $\|X_u \times X_v\|^2 = \|X_u\|^2 \|X_v\|^2 - \langle X_u, X_v \rangle^2 = EG - F^2$]

<u>Rmk</u> If S compact, then it has a well-defined area $A(S)$: cover it by regions R_i contained in local charts. S cpct $\Rightarrow$ finite # of regions $\Rightarrow$ can sum the area of all R_i's to get $A(S)$.

<u>Rmk</u> x_u, x_v lin indep $\Rightarrow$

$|\langle x_u, x_v\rangle|^2 < \|x_u\|^2 \|x_v\|^2$ by Cauchy-Schwarz

$\Rightarrow \boxed{EG - F^2 > 0}$

<u>Examples of Riemannian surfaces</u>

1) <u>$\mathbb{R}^2$</u> 2 "std" parametrizations:

- Id $\rightsquigarrow ds^2 = dx^2 + dy^2$

- Polar coord, $z : \mathbb{R}^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$,

$z(r, \theta) = (r\cos\theta, r\sin\theta)$

$\Rightarrow z_r = (\cos\theta, \sin\theta)$, $z_\theta = (-r\sin\theta, r\cos\theta)$

$\Rightarrow E(r,\theta) = \underbrace{z_r \cdot z_r}_{\text{in std } \mathbb{R}^2} = 1$, $F(r,\theta) = 0$, $G(r,\theta) = r^2$

$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2$

(2) $S \subset \mathbb{R}^3$ regular $\Rightarrow$ inner prod $\eta \cdot \xi = \eta \xi^T$ on $\mathbb{R}^3$

restricts to inner prod on T_pS

$\boxed{\text{Will always assume this inner product for regular } S \subset \mathbb{R}^3}$

Def $E\,du^2 + 2F\,du\,dv + G\,dv^2$ is called the

<u>first fundamental form</u> for $S \subset \mathbb{R}^3$ regular

Ex S graph of smooth $f: \mathbb{R}^2 \to \mathbb{R}$,
$(u,v) \mapsto f(u,v)$

$\leadsto$ $x(u,v) = (u, v, f(u,v))$ gives param. w/

$x_u = (1, 0, \underbrace{f_u}_{\frac{\partial f}{\partial u}})$, $x_v = (0, 1, \underbrace{f_v}_{= \frac{\partial f}{\partial v}})$ $\Rightarrow$

$E = \|x_u\|^2 = 1 + f_u^2$, $F = x_u \cdot x_v = f_u f_v$, $G = \|x_v\|^2 = 1 + f_v^2$

Ex $S = \mathbb{H}$, local param $\{y>0\} \subset \mathbb{R}^2 \to \mathbb{H}$

given by id.

Know $ds^2 = \frac{dx^2+dy^2}{y^2}$, which then defines

a Riemannian metric w/

$E = G = \frac{1}{y^2}$, $F = 0$

$\to A(R) = \iint_R \frac{dx\,dy}{y^2}$ $\left[\sqrt{EG-F^2} = \frac{1}{y^2}\right]$

Sim, on $S = \mathbb{D}$ get $E = G = \frac{4}{(1-x^2-y^2)^2}$, $F = 0$

$\Rightarrow A(R) = \iint_R \frac{4\,dx\,dy}{(1-x^2-y^2)^2}$