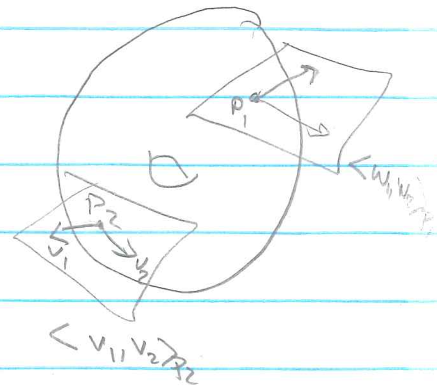
## Last time

Riemannian surface =

surface w/ a Riemannian metric




= choice of inner product $\langle , \rangle$

$\rightsquigarrow \langle , \rangle_p$ on $T_pS$, varying smoothly w/ $p$

$X : U \longrightarrow S$ local param ($U \subset \mathbb{R}^2$) $\rightsquigarrow X_u(p), X_v(p)$ basis for $T_pS$

[If $p = X(u_0, v_0)$ then
$X_u(p) = \frac{d}{dt} X(u_0 + t, v_0)\big|_{t=0}$, $X_v(p) = \frac{d}{dt} X(u_0, v_0 + t)\big|_{t=0}$]

$\rightsquigarrow E(p) = \langle X_u(p), X_u(p) \rangle_p$, $F(p) = \langle X_u(p), X_v(p) \rangle_p$, $G(p) = \langle X_v(p), X_v(p) \rangle_p$

**Today** Isometries & curvature

## §5.4 Isometries

= isomorphisms of Riemannian surfaces

= diffeomorphisms preserving the Riemannian structure:

Def A map $f: S \to S'$ between Riemannian surfaces is an isometry if

(i) $f$ a diffeo

(ii) $\underbrace{\langle df_p(v), df_p(w) \rangle_{f(p)}}_{\text{in } T_{f(p)} S'} = \underbrace{\langle v, w \rangle_p}_{\text{in } T_p S} \quad \forall p \in S, \forall v, w \in T_p S$

$f$ is a local isometry if satisfies (ii) but not (i)

((ii) $\Rightarrow df_p$ iso $\Rightarrow f$ local diffeo)

Rmk $*$ The polarization formula

$$2\langle v, w \rangle = \|v+w\|^2 - \|v\|^2 - \|w\|^2$$

$\Rightarrow$ enough to check that $\|df_p(v)\|_{f(p)} = \|v\|_p \ \forall v \in T_p S$

$*$ $\{f: S \to S ;\ f \text{ isometry}\}$ forms a gp

Q since contains id & closed under composition & inverses

$= \boxed{\text{the isometry group of } S}$

**Def 5.4.2** Intrinsic / extrinsic

| Intrinsic | Extrinsic |
|---|---|
| Properties of a Riemannian surface which are preserved under all isometries | Properties which depend on a particular representation of the surface (e.g. embedding into $\mathbb{R}^3$) |

**Ex** Arc-length & area are intrinsic

$w : [a,b] \to S$ curve w/ $\ell(w) = \int_a^b \|w'(t)\|_{w(t)} \, dt$

If $f : S \to S'$ isometry then

$$\ell(f \circ w) = \int_a^b \|(f \circ w)'(t)\|_{f \circ w(t)} \, dt = \int_a^b \| df_{w(t)}(w'(t)) \|_{f(w(t))} \, dt$$

$$\underset{f \text{ isometry}}{=} \int_a^b \|w'(t)\|_{w(t)} \, dt = \ell(w)$$

E, F, G are also "intrinsic"

**Prop 5.4.3** a) Assume

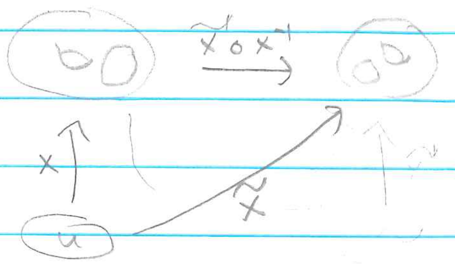
$$\begin{tikzcd} S \arrow[r, phantom, ""] & S' \\ p \in U \arrow[r, "f"] & U' \ni f(p) \\ V \subset \mathbb{R}^2 \arrow[u, "x"] \arrow[ur, "x'"'] & \end{tikzcd}$$

$\leftarrow$ local param, $x(V) = U$

f local isometry. Then $x' = f \circ x : U \to S'$ is a param.

around $f(p)$ s.t. $\underbrace{\langle x_u, x_u \rangle}_{E} = \underbrace{\langle x'_u, x'_u \rangle}_{E'}$, $\underbrace{\langle x_v, x_v \rangle}_{G} = \underbrace{\langle x'_v, x'_v \rangle}_{G'}$, $\underbrace{\langle x_u, x_v \rangle}_{F} = \underbrace{\langle x'_u, x'_v \rangle}_{F'}$

b) Conversely, suppose $x: U \to S$ & $\tilde{x}: U \to \tilde{S}$ are local param. of 2 Riemannian surfaces s.t. $E = \tilde{E}$, $F = \tilde{F}$, $G = \tilde{G}$. Then $\tilde{x} \circ x^{-1}$ is an isometry between $x(U)$ & $\tilde{x}(U)$.




Prp In Euclidean, hyperbolic & spherical geometry,

$\underbrace{\text{geometric isometries}}_{\text{preserves distance}} = \underbrace{\text{Riemannian isometries}}_{\text{preserves Riemannian metric}}$

Pf See book. □

Ex Extrinsic property: Normal v.f. of tangent planes of $S \subset \mathbb{R}^3$

But can still tell us much about the intrinsic geometry, as one sees if one studies

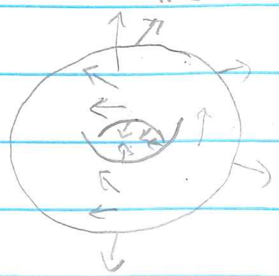
## § 5.5 Curvature

* one of the most important features in diff-geom
* some different types, all measures how the surface curves & bends

F (16) = 3

## Gaussian curvature

Let $S \subset \mathbb{R}^3$ regular surface

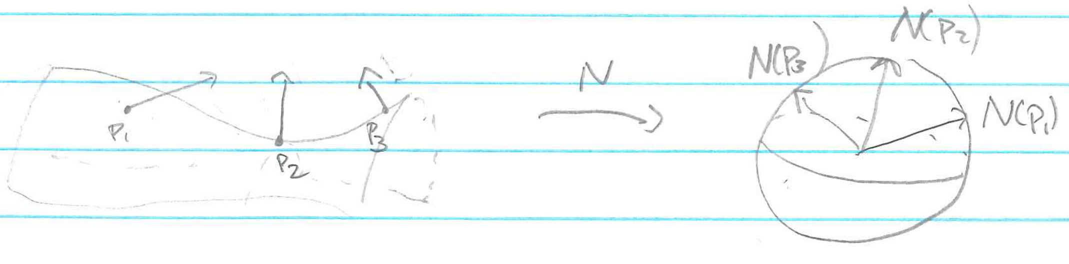


$\Rightarrow$ can use the normal v.f. of the tangent planes to see how $S$ varies

$\Rightarrow$ if $x : U \to S \subset \mathbb{R}^3$ a local param, consider ($U \subset \mathbb{R}^2$)

$N(p) = \dfrac{x_u(p) \times x_v(p)}{\|x_u(p) \times x_v(p)\|} \in \mathbb{R}^3$ normal vector of length 1

$\Rightarrow N : x(U) \longrightarrow S^2 = \{ v \in \mathbb{R}^3 \mid \|v\| = 1 \}$ the Gauss map ($x(U) \subset S$)




$S^2$ surface $\Rightarrow dN_p : T_pS \longrightarrow \underbrace{T_{N(p)} S^2}$

$\parallel \Leftarrow$ as subspaces of $\mathbb{R}^3$

$T_pS$ since $N(p)$ normal to both $T_pS$ & $T_{N(p)} S^2$

$\Rightarrow dN_p : T_pS \longrightarrow T_pS$

**Def** The <u>Gaussian curvature</u> of $S$ at $p$ is

$$K(p) = \det(dN_p).$$

**Rmk** $N(p)$ well-def up to sign $(X_u \times X_v = -X_v \times X_u)$

& $\det(d(-N_p)) = \det(-dN_p) = \det(dN_p)$

$\Rightarrow$ does not matter if we use $N$ or $-N$.

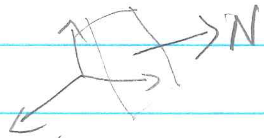
**Def** $-dN_p : T_pS \to T_pS$ the shape operator / Weingarten map

**Rmk** If $A$ a $2\times2$-matrix, then $\det(A) = \text{area}\begin{pmatrix}\text{parallelogram spanned} \\ \text{by the columns of } A\end{pmatrix}$

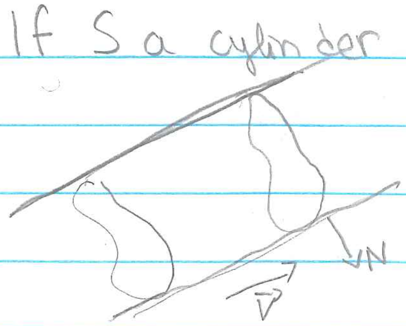
$\Rightarrow K(p)$ measures how area "changes" under the Gauss map

**Ex. 5.5.2** 1) If $S \subset \mathbb{R}^3$ a plane, then $N = \text{const}$

$\Rightarrow dN = 0 \Rightarrow K \equiv 0$



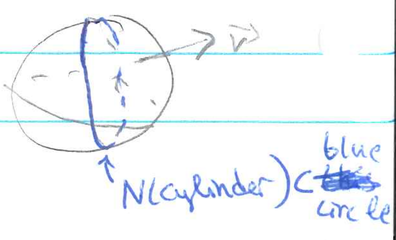

2) If $S$ a cylinder w/ axis given by $\vec{v} \in \mathbb{R}^3$




then $N(p) \in \underbrace{\text{plane w/ normal vector } \vec{v}}$

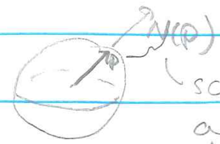
$= \text{great circle on } S^2$

$\Rightarrow \operatorname{rank}(dN) \leq 1 \Rightarrow \det(dN) \equiv 0 \Rightarrow K \equiv 0$



3) $S \subset \mathbb{R}^3$ sphere of radius $r$ centered at $0$

$\Rightarrow N(p) = \frac{p}{r}$




same dir. as $p$, but of length 1

$\Rightarrow dN_p = \frac{1}{r} I_p$, $I_p : T_pN \to T_pN$ the identity

$\Rightarrow K = \frac{1}{r^2}$ "A small sphere curves more than a big sphere"

## Gaussian curvature in local coordinates

$x : \underset{\subset \mathbb{R}^2}{U} \to S \subset \mathbb{R}^3$ local param $\Rightarrow$

$x = (x_1(u,v), x_2(u,v), x_3(u,v))$

$\Rightarrow x_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)$, $x_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right)$

$\Rightarrow$ can take further partial derivatives

$x_{uu} = \left(\frac{\partial^2 x_1}{\partial u^2}, \frac{\partial^2 x_2}{\partial u^2}, \frac{\partial^2 x_3}{\partial u^2}\right)$, $x_{uv} = \left(\frac{\partial^2 x_1}{\partial u \partial v}, \frac{\partial^2 x_2}{\partial u \partial v}, \frac{\partial^2 x_3}{\partial u \partial v}\right) = x_{vu}$ etc.

Sim, $N \circ x : U \to S^2 \subset \mathbb{R}^3$ w/ partial derivatives

$N_u = \frac{d}{dt} N(x(u+t, v))\Big|_{t=0} = dN\left(\frac{d}{dt} x(u+t, v)\Big|_{t=0}\right) = dN(x_u)$

$N_v = \frac{d}{dt} N(x(u, v+t))\Big|_{t=0} = dN(x_v)$

Recall $N \cdot x_u = 0 = N \cdot x_v$

Differentiate w.r.t $u$: $N_u \cdot x_u + N \cdot x_{uu} = 0$

$N_u \cdot x_v + N \cdot x_{uv} = 0$

$v$: $N_v \cdot x_u + N \cdot x_{uv} = 0$

$N_v \cdot x_v + N \cdot x_{vv} = 0$

Define
$$\boxed{\begin{aligned} e &= N \cdot x_{uu} = -N_u \cdot x_u \\ g &= N \cdot x_{vv} = -N_v \cdot x_v \\ f &= N \cdot x_{uv} = N \cdot x_{vu} = -N_u \cdot x_v = -N_v \cdot x_u \end{aligned}}$$

Prp 5.5.3
$$\boxed{K = \frac{eg - f^2}{EG - F^2}}$$

Pf Let $\alpha, \beta, \gamma, \delta : U \to \mathbb{R}$ s.t. $\overbrace{dN(x_u)}^{N_u} = \alpha x_u + \beta x_v$

$\underbrace{dN(x_v)}_{N_v} = \gamma x_u + \delta x_v$

Trick Take inner prod of both sides w/ $x_u$ & $x_v$:

$\Rightarrow$ $-e = N_u \cdot x_u = \alpha \overset{x_u \cdot x_u}{E} + \beta \overset{x_v \cdot x_u}{F}$

$-f = N_u \cdot x_v = \alpha F + \beta G$

$-f = N_v \cdot x_u = \gamma E + \delta F$

$-g = N_v \cdot x_v = \gamma F + \delta G$

$\Rightarrow -\begin{bmatrix} e & f \\ f & g \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ & $K(p) = \det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$\Rightarrow K(p) = \det \begin{bmatrix} e & f \\ f & g \end{bmatrix} \det \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} = \dfrac{eg - f^2}{EG - F^2}$ $\square$

<u>Def</u> $e\,du^2 + 2f\,du\,dv + g\,dv^2$ the

<u>second fundamental form</u> of S

⚠ Not intrinsic, but gives important info on how

S lies in $\mathbb{R}^3$

<u>Ex 5.5.5</u> Let $S \subset \mathbb{R}^3$ graph of smooth $h: \mathcal{O} \subset \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto h(x,y)$

$\Rightarrow S$ param by $z(x,y) = (x, y, h(x,y))$, $z_x = (1, 0, h_x)$, $z_y = (0, 1, h_y)$

$\Rightarrow E(x,y) = 1 + h_x^2(x,y)$, $G(x,y) = 1 + h_y^2(x,y)$, $F(x,y) = h_x(x,y) \cdot h_y(x,y)$

$$\Rightarrow * \quad N(x,y) = \frac{z_x \times z_y}{\|z_x \times z_y\|} = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}}$$

$* \quad z_{xx} = (0, 0, h_{xx})$, $z_{xy} = (0, 0, h_{xy})$, $z_{yy} = (0, 0, h_{yy})$

$$\Rightarrow e(x,y) = [N \cdot z_{xx}] = \frac{h_{xx}(x,y)}{\sqrt{1 + h_x^2 + h_y^2}}, \quad f(x,y) = \frac{h_{xy}(x,y)}{\sqrt{1 + h_x^2 + h_y^2}}, \quad g(x,y) = \frac{h_{yy}(x,y)}{\sqrt{1 + h_x^2 + h_y^2}}$$

$$\Rightarrow K(x,y) = \frac{h_{xx}(x,y)\,h_{yy}(x,y) - h_{xy}^2(x,y)}{\underbrace{\left((1+h_x^2)(1+h_y^2) - (h_x h_y)^2\right)}_{= 1 + h_x^2 + h_y^2}(1 + h_x^2 + h_y^2)} = \frac{\det H(h)}{[1 + h_x^2 + h_y^2]^2}$$

where $H(h)$ is the <u>Hessian</u> of h

$$H(h) = \begin{bmatrix} h_{xx} & h_{xy} \\ h_{yx} & h_{yy} \end{bmatrix}$$