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§5.6 Geodesics = "lines" in Riemannian geometry

2 ways of generalize notation of euclidean lines

(1) Straight lines are curves which minimize distance between its points.

Intrinsic! Satisfied by the hyperbolic lines! But problems on S^2/P^2.

(2) Straight lines are curves which never change direction.

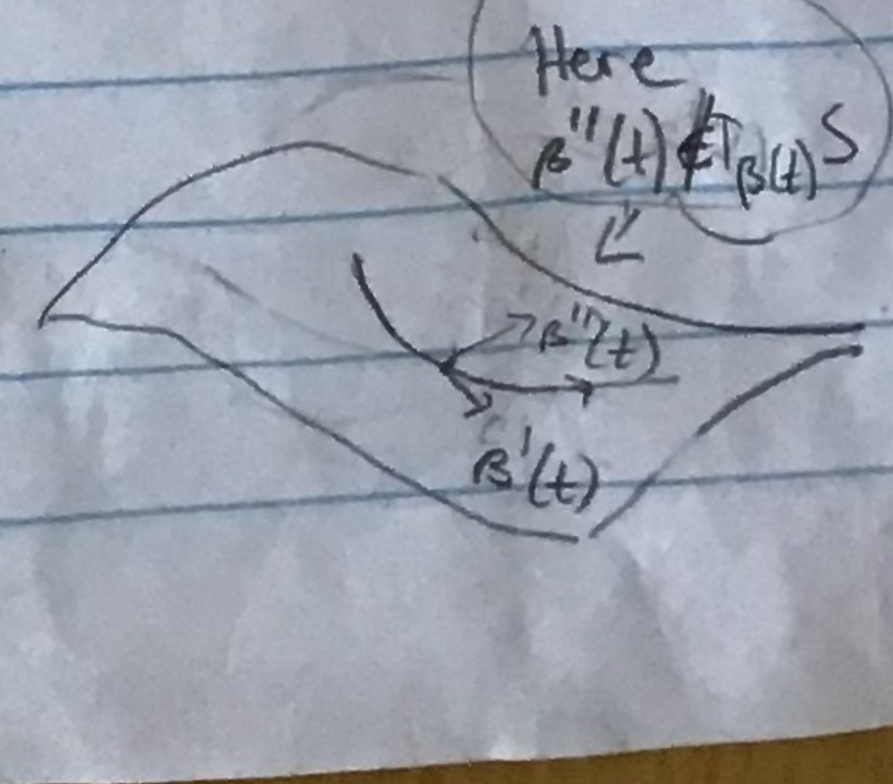
Means what ???

In $\mathbb{R}^2$: If $t \mapsto \beta(t)$ curve, then it does not change direction if $\beta'(t)$ & $\beta''(t)$ are lin dep.

(can go faster or slower, but direction is preserved $\Rightarrow$ line
ex $t \mapsto e^t(1,0)$ line w/ $\beta'(t) = (e^t, 0)$, $\beta''(t) = (e^t, 0)$
$\Rightarrow$ same direction, but $\beta'(t)$ is not constant)

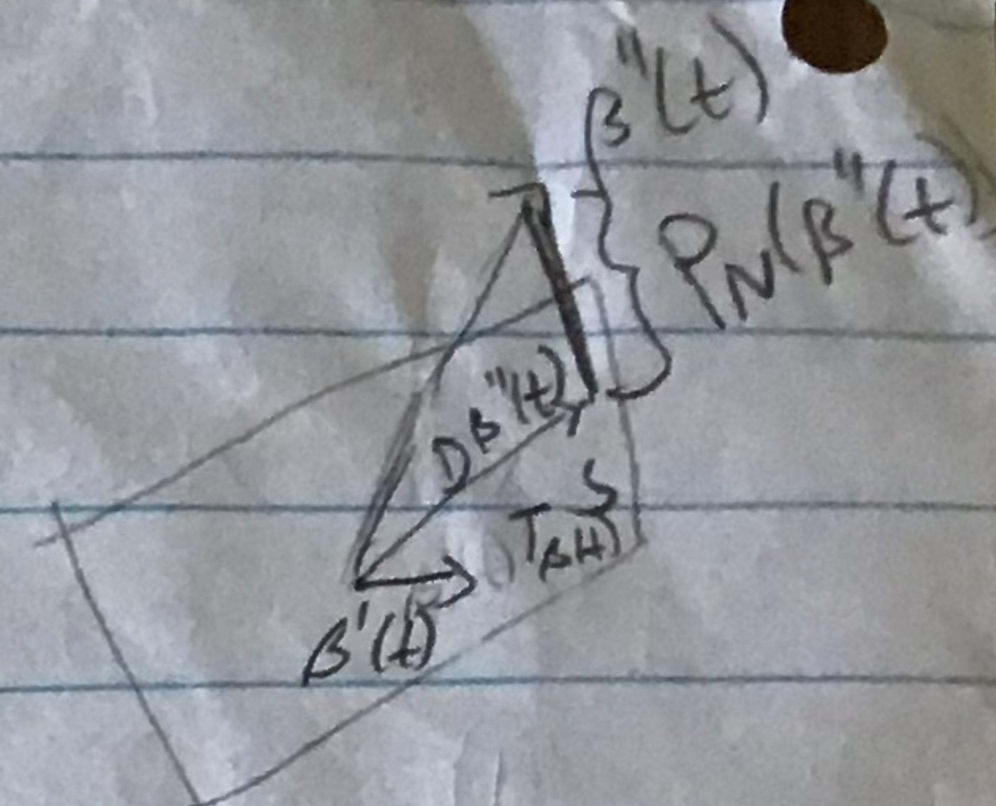
On a surface? $\beta''(t)$ is not defined!

Ex If $S \subset \mathbb{R}^3$



But $\beta''(t)$ defined if view β as a curve in $\mathbb{R}^3$. Project to $T_{\beta(t)}S$:

More precisely:



$$\underset{\mathbb{R}^3}{\beta''(t)} = \underbrace{D\beta''(t)}_{\substack{\text{orthogonal} \\ \text{proj to} \\ T_{\beta(t)}S}} + \underbrace{P_N(\beta''(t))}_{\substack{\text{orthogonal proj} \\ \text{to } N(\beta(t))}}$$

Det The component $D\beta''$ is called the <u>covariant second derivative</u> of β.

$\Rightarrow D\beta''$ "lives on S" & we can use it to define geodesics.

Def 2.6.2 The curve $t \mapsto \beta(t) \subset S$ is called a <u>geodesic</u>

if $\beta'(t) \neq 0$ & $D\beta''(t)$ is a multiple of $\beta'(t)$ $\forall t$.
($D\beta''(t) = \alpha(t)\beta'(t)$, $\alpha: I \to \mathbb{R}$ where I is the interval of def for $\beta(t)$)

Rmk The property of being a geodesic is indep. of param:

$\beta(t)$ geodesic $\Leftrightarrow$ $\beta(h(t))$ geodesic, $h: \mathbb{R} \to \mathbb{R}$ diffeo

Q Prove this.

Ex 5.6.4 (1) ℓ straight line in $\mathbb{R}^3$, $S \subset \mathbb{R}^3$ regular surface

$\Rightarrow \ell \cap S$ geodesic in S $\quad [\ell(t) = a + tv \Rightarrow \ell''(t) \equiv 0]$

(2) $\beta(t)$ geodesic $\Leftrightarrow \beta'(t)$ & $D\beta''(t)$ are lin. dep.

$$\Leftrightarrow \beta'(t) \text{ \& } \overbrace{\beta''(t) - \underbrace{P_N(\beta''(t))}_{= c(t) N(\beta(t)) \text{ for some } c(t) \in \mathbb{R}}}^{D\beta''(t)} \text{ lin dep}$$

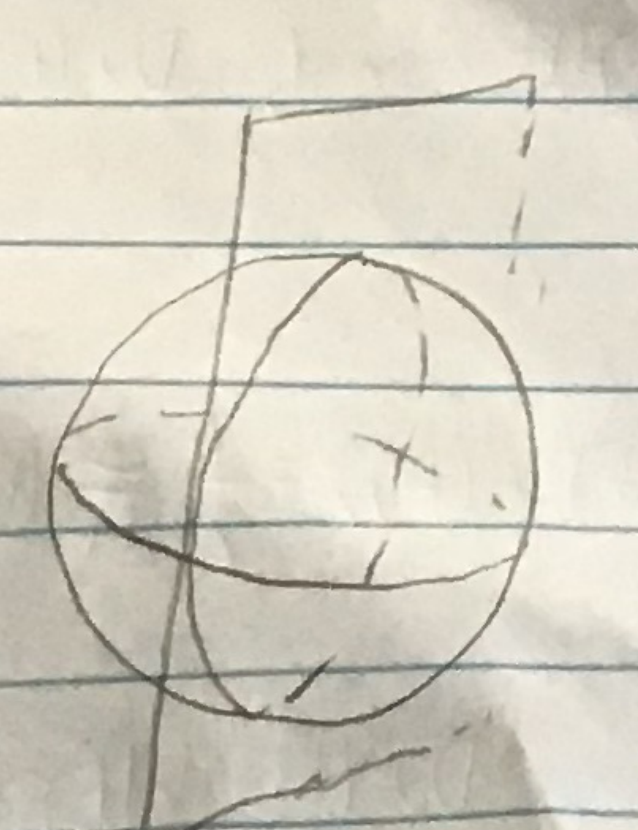
$\Leftrightarrow \beta'(t), \beta''(t)$ & $N(\beta(t))$ lin dep

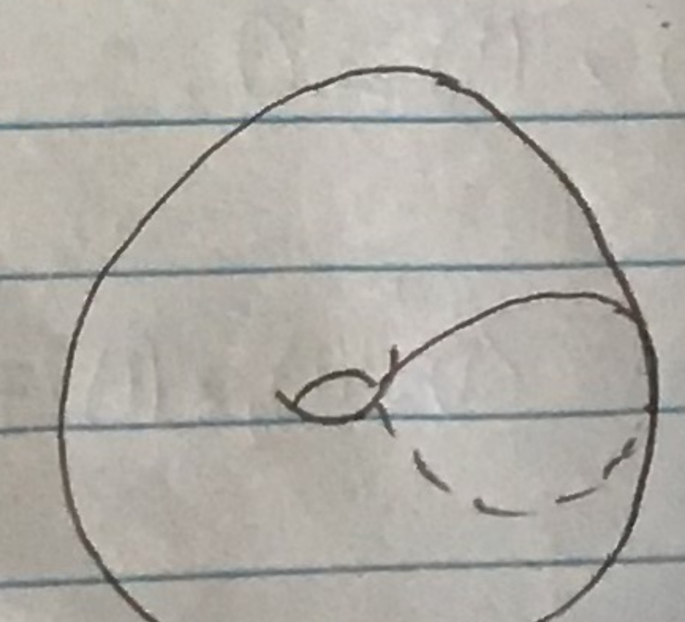
$$\Leftrightarrow \det \begin{pmatrix} \beta'(t) \\ \beta''(t) \\ N(\beta(t)) \end{pmatrix} = 0$$

A good test to check if something is a geodesic

$\Rightarrow$ if $\beta \subset S \cap P$ where P plane normal to S along $S \cap P$

then $\beta(t)$ a geodesic (since then $\beta'(t), \beta''(t)$ & $N(\beta(t)) \in P$ $\forall t$)

$\Rightarrow$  great circles

 $\leftarrow$ generating curves of surfaces of revolution

are geodesic.

Recall Being a geodesic is indep. of param.

$\Rightarrow$ can choose β to be param. by arc-length;

Def A curve $s \mapsto \alpha(s)$, $s \in I$, is

* parametrized by arc-length if $\|\alpha'(s)\| = 1 \ \forall s \in I$

* a constant speed curve if $\|\alpha'(s)\| = \text{const} \ \forall s \in I$

Prp Any regular curve can be parametrized by arc-length

Pf Let $\beta(t)$, $t \in [a,b]$ be a regular curve, let

$$s(t) = \int_a^t \|\beta'(\tau)\| \, d\tau \Rightarrow \frac{ds}{dt} = \|\beta'(t)\|$$

$\beta'(t) \neq 0 \ \forall t \Rightarrow s(t)$ invertible $\Rightarrow \alpha(s) = \beta(t(s))$ arc-length param of β. $\square$

Prp 5.6.5 A curve param. by $\alpha(t)$ is a constant speed geodesic
$\|\alpha'(t)\| = \text{const}$, $D\alpha''(t) = k(t)\alpha'(t)$

$\Leftrightarrow D\alpha''(t) \equiv 0$.

Pf Note $\langle D\alpha''(t), \alpha'(t)\rangle = \alpha''(t) \cdot \alpha'(t) = \frac{1}{2}\frac{d}{dt}\langle \alpha'(t), \alpha'(t)\rangle = \frac{1}{2}\frac{d}{dt}\|\alpha'(t)\|^2 \ (*)$

"$\Leftarrow$" $D\alpha''(t) \equiv 0 \Rightarrow \|\alpha'(t)\| = \text{const}$ by $(*)$ & clearly $D\alpha''(t) = \underbrace{k(t)}_{=0}\alpha'(t)$
$\Rightarrow \alpha(t)$ constant speed geodesic

"$\Rightarrow$" $\overbrace{\|\alpha(t)\| - c}^{\text{Given}}$ & $D\alpha''(t) = k(t)\alpha'(t)$. Prove $k(t) = 0$. But

$$k(t) = \langle D\alpha''(t), \alpha'(t)\rangle \overset{(*)}{=} \frac{1}{2}\frac{d}{dt}\|\alpha'(t)\|^2 \overset{\text{assumption}}{=} 0$$

$\square$

<u>Ex</u> $S = \{x \in \mathbb{R}^3 ; |x| = R\} \subset \mathbb{R}^3$ sphere of radius R

If $s \mapsto \alpha(s)$ curve on S param by arc-length

$$\Rightarrow \quad D\alpha'' = \alpha'' - \underbrace{\frac{\alpha''(s)\cdot\alpha(s)}{R^2}\,\alpha(s)}_{\text{proj to normal}}$$

and $\alpha''(s)\cdot\alpha(s) = \underbrace{(\alpha'(s)\cdot\alpha(s))'}_{=0} - \underbrace{|\alpha'(s)|^2}_{=1} = -1$

$\Rightarrow$ α constant speed geodesic $\Leftrightarrow$

$$\alpha''(s) + \frac{1}{R^2}\alpha(s) = 0$$

$\Rightarrow \alpha(s) = A\cos(s/R) + B\sin(s/R)$, $A, B \in \mathbb{R}^3$

$\alpha(s) \in S \ \forall s \Leftrightarrow |\alpha(s)| = R \ \forall s$

$\Rightarrow |A| = |B| = R$, $\langle A, B\rangle = 0$

$R^2 = \|\alpha(s)\|^2 = \langle A\cos(s/R) + B\sin(s/R), A\cos(s/R) + B\sin(s/R)\rangle$

$= \cos^2(s/R)\|A\|^2 + \sin^2(s/R)\|B\|^2 + 2\cos(s/R)\sin(s/R)\langle A, B\rangle$

$\rightarrow \alpha(s)$ great circle on S

$D\alpha''(s)$ in Local coordinates – Cristoffel symbols

Cristoffel symbols

$S \subset \mathbb{R}^3$ regular, $\quad X : U \longrightarrow S$ Local param. ($U \subset \mathbb{R}^2$)

Let $\Gamma_{ij}^k : U \to \mathbb{R}$, $i,j,k = 1,2$, be defined by

$$X_{uu}(p) = \Gamma_{11}^1(p) X_u(p) + \Gamma_{11}^2(p) X_v(p) + e(p) N(p)$$

$$X_{uv}(p) = \Gamma_{12}^1(p) X_u(p) + \Gamma_{12}^2(p) X_v(p) + f(p) N(p)$$

$$X_{vv}(p) = \Gamma_{22}^1(p) X_u(p) + \Gamma_{22}^2(p) X_v(p) + g(p) N(p)$$

("coefficients" when we write X_{uu}, X_{uv}, X_{vv} in terms of the basis (X_u, X_v, N))

These are the Cristoffel symbols of S w.r.t. the parametrization X.

Lemma A The Cristoffel symbols are intrinsic.

Pf From the pf of Theorema egregium it follows that

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v & F_v - \frac{1}{2}G_u \\ F_u - \frac{1}{2}E_v & \frac{1}{2}G_u & \frac{1}{2}G_v \end{bmatrix}$$

(See book, p 141) $\square$

They are useful for finding geodesics:

Thm 5.6.9

(i) Suppose $\alpha(s) = x(u(s), v(s))$ is a param. of a smooth curve on S. Then α is a constant speed geodesic $\Longleftrightarrow$ $u(s)$ & $v(s)$ satisfy the following system of differential eqns

$$\left.\begin{aligned} u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 = 0 \\ v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2 = 0 \end{aligned}\right\} \text{The Geodesic eqns}$$

(ii) Geodesics are preserved by isometries

Pf (i) $\alpha'(s) = u'x_u + v'x_v$ in local param $x: U \to S \subset \mathbb{R}^3$, $U \subset \mathbb{R}^2$

$$\Rightarrow \alpha''(s) = u''x_u + v''x_v + (u')^2 x_{uu} + 2u'v'x_{uv} + (v')^2 x_{vv}$$

"project away" N-dir

$$\Rightarrow D\alpha''(s) = u''x_u + v''x_v + (u')^2(\Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v)$$
$$+ 2u'v'(\Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v) + (v')^2(\Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v)$$

$$\Rightarrow D\alpha''(s) = 0 \Leftrightarrow \begin{cases} u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 = 0 \\ v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2 = 0 \end{cases}$$

iii) Follows from

Thm 5.6.7 The covariant second derivative

is intrinsic:

If $\Phi : S \to S'$ is an isometry between

regular surfaces, then

$$d\Phi_{\beta(t)}(D\beta''(t)) = D(\Phi\beta)''(t).$$

+ Lma A $\square$

Ex 5.6-10 (3) Find all geodesics in the upper half-plane

model of the hyperbolic plane:

Use coord $z = x + iy$. Know $E = G = 1/y^2$, $F = 0$

$\Rightarrow E_y = G_y = \frac{-2}{y^3}$, $E_x = G_x = F_x = F_y = 0$

Calculate the Cristoffel symbols

$$\begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{y^3} & 0 \\ \frac{1}{y^3} & 0 & -\frac{1}{y^3} \end{bmatrix}$$

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$\Rightarrow \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0, \quad \Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$

$\Rightarrow$ The Geodesic equations are given by:

$$x'' - \frac{2x'y'}{y} = 0 \quad (i), \qquad \begin{cases} x'' + (x')^2 \cdot 0 + 2x'y' \cdot \left(-\frac{1}{y}\right) + (y')^2 \cdot 0 = 0 \\ y'' + (x')^2 \cdot \frac{1}{y} + 2x'y' \cdot 0 + (y')^2 \cdot \left(-\frac{1}{y}\right) = 0 \end{cases}$$

$$y'' + \frac{(x')^2}{y} - \frac{(y')^2}{y} = 0 \quad (ii)$$

2 cases:

* $x'(s) = 0 \Rightarrow x(s) = a$ gives a sol for $a \in \mathbb{R}$ since then

(i) is triv. satisfied

$(ii): \; y'' - \frac{(y')^2}{y} = 0 \quad \Leftrightarrow \quad \left(\frac{y'}{y}\right)' = 0 \qquad [\text{know } y > 0]$

$\Rightarrow y = be^{cs}$ for some $b, c \in \mathbb{R}$

α param by arc-length $\Rightarrow c = 1$

$\Rightarrow \alpha(s) = a + be^{s} i$, which is a param of a vertical $\mathbb{H}$-line

$\Rightarrow$ all vertical $\mathbb{H}$-lines are geodesic

* $x'(s) \neq 0$: Then (i) is separable

$\left(\int \frac{x''}{x'} = \int \frac{2y'}{y} \Rightarrow \ln|x'| = 2\ln y + \tilde{C} \Rightarrow\right)$

$\frac{x''}{x'} = \frac{2y'}{y}$ integrate $\rightsquigarrow x' = Cy^2, \; C \neq 0$

<u>Trick</u> Use arc-length param:

$$(x')^2 \underbrace{E}_{=\frac{1}{y^2}} + 2\underbrace{F}_{=0} x'y' + (y')^2 \underbrace{G}_{=\frac{1}{y^2}} = 1$$

$$\Rightarrow \underbrace{(x')^2}_{cy^2} + (y')^2 = y^2 \qquad \Rightarrow y' = \pm y\sqrt{1-c^2y^2}$$

Mult. by $\pm \frac{cy}{\sqrt{1-c^2y^2}}$ $\Rightarrow$ $\frac{cyy'}{\sqrt{1-c^2y^2}} = \underbrace{cy^2}_{=x'}$

Integrate: $\underbrace{\int \frac{cyy'}{\sqrt{1-c^2y^2}}\,dt}_{\frac{1}{c}\sqrt{1-c^2y^2}} = \int \underbrace{x'}_{x-m \text{ for some } m\in\mathbb{R}}$

$$\Rightarrow (x-m)^2 = \frac{1-c^2y^2}{c^2} \quad \Leftrightarrow \underbrace{(x-m)^2 + y^2 = \frac{1}{c^2}}_{\text{circle centered on x-axis}}$$

$\Rightarrow$ this gives also a $\mathbb{H}$-line

$\Rightarrow$ all geodesics are contained in $\mathbb{H}$-lines