Last time

Geodesics = "straight lines" on $S \subset \mathbb{R}^3$

$\Leftrightarrow \beta'(t)$ & $\underbrace{D\beta''(t)}_{\substack{\perp \text{ proj of} \\ \beta''(t) \text{ to } T_{\beta(t)}S \\ \cap \\ \mathbb{R}^3}}$ parallel $\forall t$

$\beta(t)$ $\underbrace{\text{constant speed}}_{\|\beta'(t)\| = \text{const}}$ geodesic $\Longleftrightarrow D\beta''(t) = 0$

local param $x : U \xrightarrow{\subset \mathbb{R}^2} S$

$\Longleftrightarrow \beta(t) = x(u(t), v(t))$ sol. to the Geodesic equations

$$\left.\begin{aligned} u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1 = 0 \\ v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2 = 0 \end{aligned}\right\} (*)$$

Γ_{ij}^k the Christoffel symbols def by

$$x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + eN$$
$$x_{uv} = \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + fN$$
$$x_{vv} = \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + gN$$

Rmk In general very difficult to solve $(*)$ explicitly.

But (*) is a second order ODE

(ordinary differential eqn)

Picard-Lindelöf Thm

$\Rightarrow$ if we specify initial cond $u(0), v(0), u'(0), v'(0)$ it will

have a unique sol defined on some time interval, and the sol. varies smoothly w/ the initial conditions $\Rightarrow$

<u>Prp 5.6.11</u> Every point of a Riemannian surface S

has an open nbhd V s.t.:

$\exists\ \varepsilon, \tau > 0$ s.t. $\forall q \in V$ & $w \in T_qS$ w/ $\|w\| < \varepsilon$,

there is a unique constant speed geodesic

$\gamma_w^q : (-\tau, \tau) \to S$ s.t. $\underbrace{\gamma_w^q(0) = q, \quad (\gamma_w^q)'(0) = w}_{\text{initial conditions}}$

Moreover, $\gamma_w^q(t)$ depends smoothly on q, w & t.

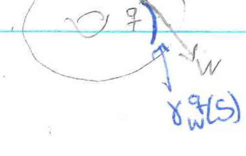
<u>Rmk</u> (*) homogeneous (does not depend explicitly on t)

$\Rightarrow$ if $\gamma(t)$ is a sol, then $\eta(t) = \gamma(ct)$ is also a sol.

w/ initial cond $\eta(0) = \gamma(0)$, $\eta'(0) = c\gamma'(0)$

$\Rightarrow \gamma_{cw}^q(t) = \gamma_w^q(ct)$ (**)

$\Rightarrow$ may assume $\tau > 1$



⇝ The exponential map

Let $B_q(\varepsilon) = \{ w \in T_q S ; \|w\| < \varepsilon \}$, ε from Prp 5.6.11.

Def The map $\exp_q : B_q(\varepsilon) \to S$, $\exp_q(w) = \gamma^q_w(1)$

is called the exponential map

Rmk $\exp_q(sw) = \gamma^q_{sw}(1) = \gamma^q_w(s)$ by (**)

$\Rightarrow$ $s \mapsto \exp_q(sw)$ is a geodesic for each $w \in B_q(\varepsilon)$

& $\frac{d}{ds} \exp_q(sw) \Big|_{s=0} = w$ (motivates the name)

These are the most important properties of the exponential map:

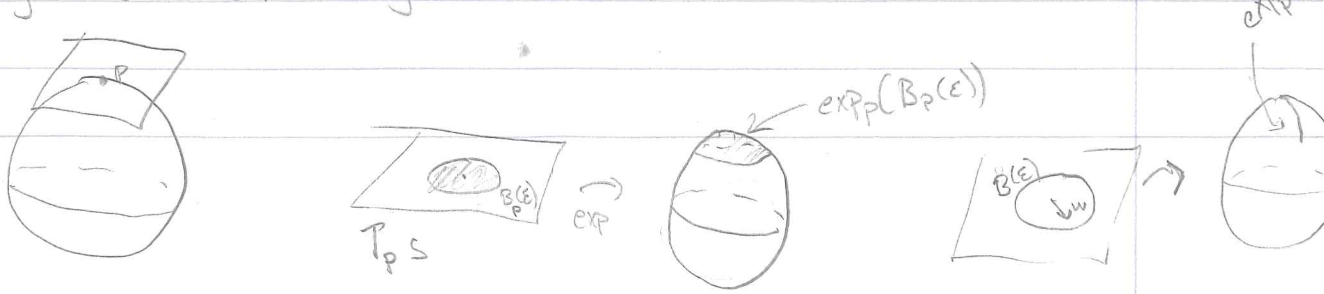
Thm 5.6.12 For every $p \in S$ there is an $\varepsilon > 0$ s.t.

(1) $\exp_p$ is a diffeo between $B_p(\varepsilon)$ & $\underbrace{\text{a nbhd } V \text{ of } p}_{\overset{\text{def}}{=} \text{ a normal nbhd}}$

(2) If ε is small enough, any 2 pts in

$\exp_p(B_p(\varepsilon))$ can be joined by a unique

geodesic of length $< 2\varepsilon$.



From this we get local parametrizations of S $\rightsquigarrow$

§5.7 Geodesic polar coordinates

Given $p \in S$, assume we have fixed an ON-basis $(\text{w.r.t. } \langle , \rangle_p)$ of T_pS $\rightsquigarrow$ cartesian coord (u,v) on T_pS

$\rightsquigarrow$ polar coord (r,θ) on T_pS s.t.

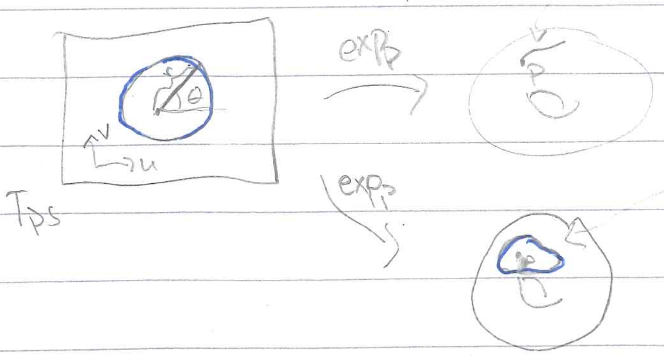
$(u,v) = (r\cos\theta, r\sin\theta) \Rightarrow$ get local param

$$x : \underset{T_pS}{\underset{\cap}{B_p(\varepsilon)}} \longrightarrow S, \quad x(r,\theta) = \exp_p((r\cos\theta, r\sin\theta)),$$

$$\underbrace{(r,\theta)}_{\boxed{\text{Geodesic polar coordinates}}} \in (-\varepsilon,\varepsilon) \times \underbrace{J}_{\substack{\text{interval of length} < 2\pi \\ \text{(to get 1-1)}}}$$

Def $r = \text{const} \Rightarrow \exp_p((r\cos\theta, r\sin\theta))$ geodesic circle

$\theta = \text{const} \Rightarrow$ geodesic radii



the coordinate curves for geodesic polar coord.

⚠ Geodesic radii are geodesics,

⚠ geodesic circles are usually not

Rmk Geodesic radii constant speed geodesics,

w.l.o.g assume they are param by arc-length.

$\Rightarrow$ for the metric $ds^2 = E(r,\theta)dr^2 + F(r,\theta)dr\,d\theta + G(r,\theta)d\theta^2$

we always have $\underset{\langle x_r, x_r\rangle}{E(r,\theta)} = 1$

Lma 5.7.2 ((Gauss' lma) $F(r,\theta) = 0$ for (r,θ) geodesic polar coord.

Pf $Dx_{rr} = 0$ since $r \mapsto x(r,\theta)$ w/ θ fixed are

constant speed geodesics.

But $Dx_{rr} = \Gamma_{11}^1 x_r + \Gamma_{11}^2 x_\theta \Rightarrow \Gamma_{11}^1 = \Gamma_{11}^2 = 0$

$\Rightarrow 0 = \Gamma_{11}^1 F + \Gamma_{11}^2 G = F_r - \frac{1}{2}E_\theta$

from formula for the Christoffel symbols

But $E \equiv 1 \Rightarrow E_\theta = 0 \Rightarrow F_r = 0$

$\Rightarrow F(r,\theta) = F(0,\theta) = \langle x_r(0,\theta), \underbrace{x_\theta(0,\theta)}_{=0} \rangle = 0$ $\square$

Ex 5.7.1 & 5.7.3 Geodesic polar coord & metric for $E^2, S^2, \mathbb{D}$.

(1) E^2 : At $p=0$; $x(r,\theta)=(r\cos\theta, r\sin\theta)$

Metric $G=\langle x_\theta, x_\theta\rangle = r^2 \Rightarrow ds^2 = dr^2 + r^2 d\theta^2$

2) S^2: At $p=(0,0,1)$

Know Geodesics are given by $\gamma(s)=A\cos(s)+B\sin(s)$

w. initial cond $\gamma(0)=A$, $\gamma'(0)=B$

$\Rightarrow A=(0,0,1)$, $\gamma(s)=\exp_p(sB)$

and since $T_pS \simeq \mathbb{R}^2\times\{0\} = \{(w_1,w_2,0)\in\mathbb{R}^3\} \subset \mathbb{R}^3$

we get $\exp_p(r\underbrace{(\cos\theta,\sin\theta)}_{B}) = (0,0,1)\cos r + (\cos\theta,\sin\theta,0)\sin r$

$\Rightarrow \boxed{x(r,\theta)=(\sin r\cos\theta, \sin r\sin\theta, \cos r)}$

Metric $G=\langle x_\theta, x_\theta\rangle = \langle(-\sin r\sin\theta, \sin r\cos\theta, 0), (-\sin r\sin\theta, \sin r\cos\theta, 0)\rangle$

$= \sin^2 r\sin^2\theta + \sin^2 r\cos^2\theta = \sin^2 r$

$\Rightarrow ds^2 = dr^2 + \sin^2 r\, d\theta^2$

$\mathbb{D}$: $p = 0$. Know geodesics are radii

Q Why?

$\Rightarrow x(r,\theta) = (h(r)\cos\theta, h(r)\sin\theta)$ for some $h(r)$

Some different ways of finding $h(r)$

1) $r \mapsto (\underbrace{h(r)\cos\theta}_{u}, \underbrace{h(r)\sin\theta}_{v})$ should be param.

by arc-length $\Rightarrow$

$$\underbrace{E(u,v)\left(\frac{du}{dr}\right)^2 + F(u,v)\frac{du}{dr}\frac{dv}{dr} + G(u,v)\left(\frac{dv}{dr}\right)^2}_{} = 1$$

$$= \frac{4(h'(r))^2\cos^2\theta + 0 + 4(h'(r))^2\sin^2\theta}{(1-u^2-v^2)^2}$$

$\Rightarrow 4(h'(r))^2 = (1-h^2(r))^2$. Solve this

Hint $\tanh x$ satisfies $1-\tanh^2 x = (\tanh x)'$

2) Solve the geodesics eqn in $\mathbb{D}$ under the assumption that a sol is given by $(u(r), 0)$ + use rotation symmetry.

3) $r \mapsto (h(r)\cos\theta, h(r)\sin\theta)$ should be param. by arc-length $\Rightarrow$ the hyperbolic length of $(h(r)\cos\theta, h(r)\sin\theta)$ should be given by r.

and we recall from what we did before that a ray from the origin in $\mathbb{D}$ of hyperbolic length r has euclidean length $\tanh \frac{r}{2}$ $\Rightarrow$

$h(r) = \tanh \frac{r}{2}$.

Metric $\quad X_\theta = (\underbrace{-\tanh\tfrac{r}{2}\sin\theta}_{a}, \underbrace{\tanh\tfrac{r}{2}\cos\theta}_{b})$

$$\Rightarrow \langle X_\theta, X_\theta \rangle = 4\,\frac{a^2+b^2}{(1-u^2-v^2)^2} = \frac{4\tanh^2(r/2)}{(1-\tanh^2(r/2))^2} = \sinh^2(r)$$

$$\Rightarrow ds^2 = dr^2 + \sinh^2 r \, d\theta^2$$