Betweenness axioms

Notation $A*B*C$ → B lies between A & C

Q What would this mean in incidence geometry w/ only 3 or 4 points? 5?

The axioms:

B1 If $A*B*C$, then A, B, C are distinct points on a line, and $C*B*A$ also holds.

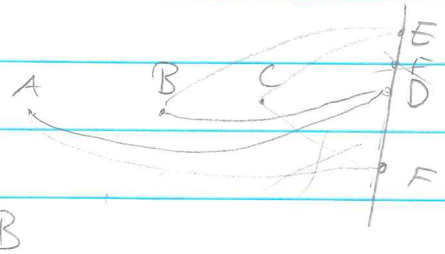
B2 Given 2 distinct points A and B, there exists a point C such that $A*B*C$.

B3 If A, B and C are distinct points on a line, then one and only one of the relations $A*B*C$, $B*C*A$ and $C*A*B$ is satisfied.

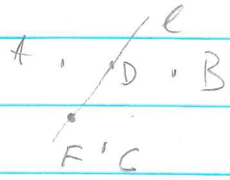
B4 Let A, B and C be points not on the same line and let l be a line which contains none of them. If $D \in l$ and $A*D*B$, there exists an E on l s.t. $B*E*C$, or an F on l s.t.

$A*F*C$, but not both.

Q Draw B4



open mind!



Rmk B4 can also be formulated: (Pasch's axiom)

"If a line l goes through a side of a triangle but none of it vertices, then it also goes through (exactly) one of the other sides"

consequence of the other axioms, Exc 6.

Q Ex on when B4 does not hold?

A $\mathbb{R}^n$, $n>2$

$\Rightarrow$ (I3 + B4) $\Rightarrow$ the geometry is 2-dim.

Task Define betweenness:

In standard Euclidean plane ($\mathbb{E}^2$) (= the v.sp. $\mathbb{R}^2$ w/ inner product)

can use distance:

$A * B * C \iff A, B, C$ are distinct &

$$d(A,C) = d(A,B) + d(B,C)$$

($d(X,Y)$ = distance between X & Y)

(Q Exc Is B1 - B4 satisfied?)

B2 = triangle ineq $\underbrace{\|x+y\|}_{A-C} \le \underbrace{\|x\|}_{A-B} + \underbrace{\|y\|}_{B-C}$

(Exc 3 This is also ok for $\mathbb{Q}^2$, but not $\mathbb{Z}^2$.)

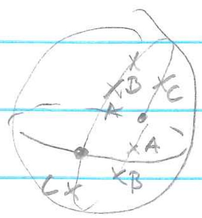
Ex $K \subset \mathbb{R}^2$ open, convex w/

$\{\text{lines}\} = \{\ell \cap K : \ell \text{ line in } \mathbb{R}^2, \ell \cap K \neq \emptyset\}$

Betweenness defined as in $\mathbb{R}^2$

Q Is this ok?

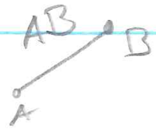
Non-ex $\mathbb{P}^2$, since lines = great circles/antipodal points



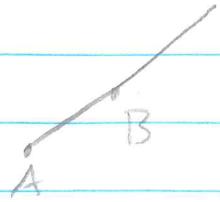
$\simeq$ circles

Consequences of B1 - B4:

Def 1 The segment $AB = \{A, B\} \cup \{C : A * C * B\}$



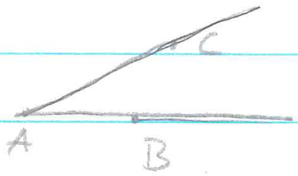
Def 2 The ray $\overrightarrow{AB} = AB \cup \{C \mid A*B*C\}$



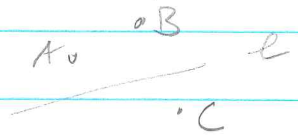
Rmk $\overline{AB} = \overrightarrow{AB} \cup \overrightarrow{BA}$

Def 3 Assume A, B, C not on a line. Then the <u>angle</u>

$$\angle BAC = \{\overrightarrow{AB}, \overrightarrow{AC}\}$$

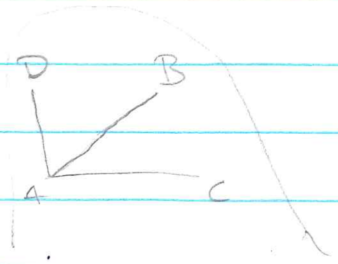


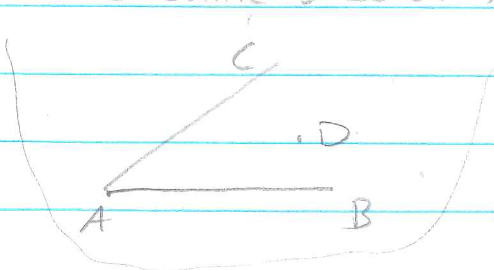
Def 4 Let ℓ be a line. Then two points A, B are <u>on the same side of ℓ</u> if $AB \cap \ell = \emptyset$



Def 5 A point D is <u>inside</u> the angle $\angle BAC$ if

- B and D are on the same side of $\overline{AC}$
- C and D are on the same side of $\overline{AB}$





Def 6 The angles $\angle BAC$ and $\angle BAD$ are on the same (resp. opposite) side of $\overrightarrow{AB}$ if C & D are on the same (resp. opposite) side of $\overline{AB}$

⚠ For the last def. to make sense we need to define what "opposite side" of a line means →

<u>Exc 4</u> There are exactly 2 sides of a line l.

Def 5 ⇒ we can talk about points <u>inside</u> and <u>outside</u> a triangle

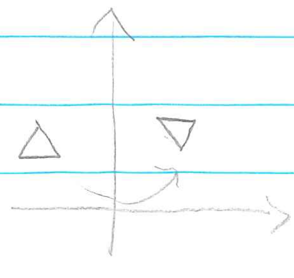
<u>Q</u> Define this.

Let's go on to the

<u>Axioms of congruence</u>

1* Needed to be able to talk about 2 different configuration "as being the same"

* <u>Intuitively</u>: 2 different configurations should be the same if ∃ a rigid motion taking one into the other



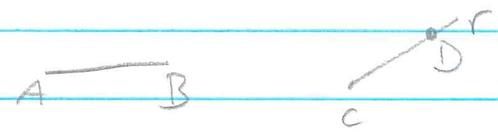
* In $\mathbb{E}^2$: can be defined in terms of angle measures & distance.

To define it abstractly:

<u>Notation</u> $\cong$ to mean congruent

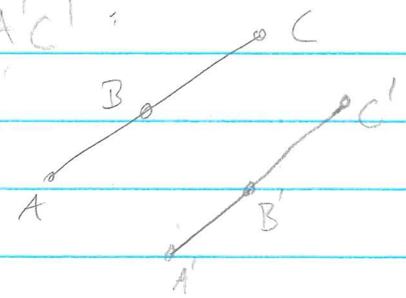
<u>Axioms for congruence of segments:</u>

<u>C1</u> Given a segment AB and a ray r from C, there is a uniquely determined point D on r s.t. $CD \cong AB$



<u>C2</u> $\cong$ is an equivalence relation on the set of segments

<u>C3</u> If $A*B*C$ and $A'*B'*C'$ and both $AB \cong A'B'$, $BC \cong B'C'$ then also $AC \cong A'C'$:



<u>Rmk</u> If we have a distace fcn d & this is used to define $A*B*C$, then can define $AB \cong A'B'$ as $d(A,B) = d(A',B')$

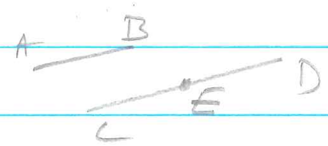
<u>Q</u> Check this is ok

<u>Consequences of C1-C3</u>

- Can define "size" (even w/o distance function):

<u>Def</u> AB is <u>shorter</u> than CD ($AB < CD$) if

$\exists$ point E s.t. $C * E * D$ and $AB \cong CE$



- Can define circle:

<u>Def</u> Given a point O and a segment AB, define the

<u>circle w center O and radius AB</u> $= \{C \in S \mid OC \cong AB\}$

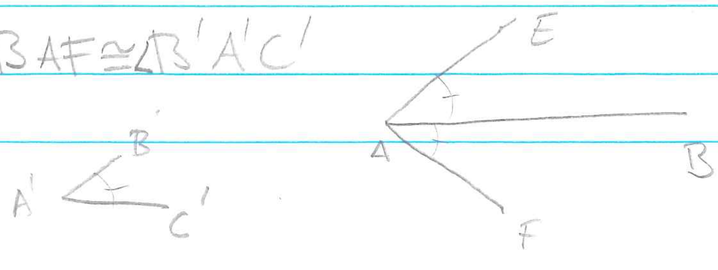
<u>Q</u> Convince yourself that is nonempty.

<u>Axioms for congruences of angles</u>:

<u>C4</u> Given a ray $\overrightarrow{AB}$ and an angle $\angle B'A'C'$, there
($\sim C1$)

are angles $\angle BAE$ and $\angle BAF$ on opposite sides of

$\overline{AB}$ s.t. $\angle BAE \cong \angle BAF \cong \angle B'A'C'$



C5 $\cong$ equivalence relation on the set of angles
(~C2)

C6 Given triangles ABC and $A'B'C'$, If $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$, then the two triangles are congruent - i.e. $BC \cong B'C'$

$$\angle ABC \cong \angle A'B'C'$$

$$\angle BCA \cong \angle B'C'A'$$

RMK C6 = SAS = side-angle-side congruence criterion

[a triangle is determined up to congruence by an angle & its adjacent sides]

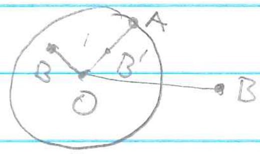
Ex In $\mathbb{E}^2$ congruence is defined as equivalence under the action of the Euclidean gp = gp of transformations generated by rotations & translations = {transformations that preserve distances}

Q Check that C1–C6 hold in this case

2 more axioms needed to get "the same" geometry as Euclid (to be able to prove all properties from "The Elements")

A continuity axiom:

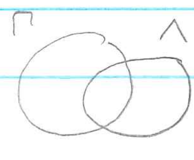
Def Let Γ be a circle w/ center O & radius OA. Then a point B is inside Γ if $OB < OA$ and outside if $OA < OB$.

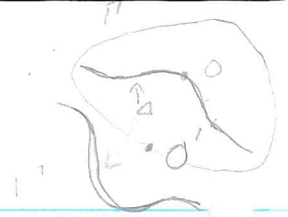


Def A line ℓ (or circle Γ') is tangent to the circle Γ if $\ell \cap \Gamma = \{P\}$ ($=\{$one point$\}$)
($\Gamma' \cap \Gamma = \{P\}$)

Hilbert's axiom E:

Given 2 circles Γ and Δ such that Δ contains points both inside and outside Γ. Then Γ and Δ have common points.

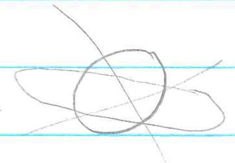




<u>Rmk</u> We express this as "they intersect".

$\tilde{Q}$ Check that this do not follow from the other axioms

$\tilde{Q}$ Other axioms $\Rightarrow$ $\Gamma \cap \Delta = \{P_1, P_2\}$

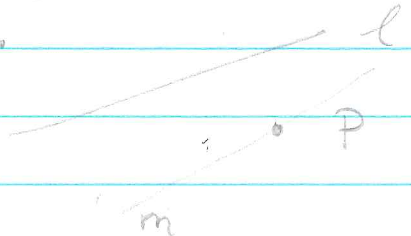


A variant:

<u>E'</u> If a line ℓ contains points both inside and outside the circle Γ, then ℓ & Γ will intersect.

Finally, the <u>Axiom of parallels</u>:

<u>P</u> (Playfair's axiom, 1795) Given a line ℓ and a point P not on the line. Then there is at most one line m through P which does not intersect ℓ.



<u>Def</u> If two lines ℓ, m do not intersect, we say that they are <u>parallel</u>, $m \parallel \ell$.

$\tilde{Q}$ Show that the <u>existence</u> of m follows from the other axioms

$\Rightarrow$ the uniqueness is the important thing here.

Rmk * Goes back to Proclus in 5:th century

* The most famous axiom, since cool things happens if we remove it. Start explore this in the next lecture

* $\mathbb{E}^2$ satisfies all these axioms.

But also $\mathbb{F}^2$, $\mathbb{F}$ any ordered field where every element has a square root.

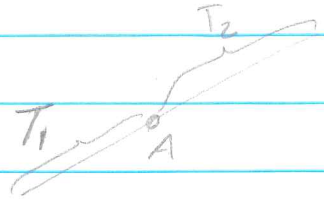
See in the book how one can replace axiom E w/ another axiom D so that I^*, B^*, C^*, D & P determine $\mathbb{E}^2$ unique

To get axiom system that only works for $\mathbb{E}^2$, can replace axiom E w/

D [Dedekind's axiom] If a line l is a disjoint union of 2 subsets T_1 & T_2 s.t. all the points of T_1 are on the same side of T_2 and vice versa, then there is a unique point $A \in l$ s.t. if

$B_1 \in T_1$, $B_2 \in T_2$ then either $A = B_1$, $A = B_2$ or

$B_1 * A * B_2$



($\approx T_1 \cap T_2 = A$)

A <u>completeness axiom</u> – connected to Dedekind's def of $\mathbb{R}$

consequence: the geometry on any line $\approx$ geometry on $\mathbb{R}$

$I^*, B^*, C^*, D + P$ determine Euclidean geometry completely

stop!

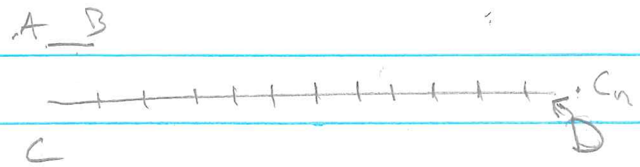
$D \Rightarrow E$

$D \Rightarrow \underline{A}$ (Axiom of Archimedes)

Given 2 segments AB and CD, there are points

$C = C_0, \dots, C_n$ on $\overrightarrow{CD}$ s.t. $C_i C_{i+1} \cong AB$ for

every $i < n$ and $CD < CC_n$



<u>Consequences</u> A geometry w/ axioms I^*, B^*, C^*, P, E & A

can be identified w/ a subset of $\mathbb{E}^2$