Last lecture! Remains:

Today * Geodesics are the "shortest" curves

* Geodesically complete surfaces

* The Gauss-Bonnet theorem

Last time Continued to study geodesics.

* $\beta(t) \subset S$ geodesic if $\underbrace{D\beta''(t)}$ & $\beta'(t)$ lin dep

defined this for $S \subset \mathbb{R}^3$, but have seen Christoffel symbols are intrinsic

$\Rightarrow$ can define $D\beta''(t)$ on any Riemannian surface by:

$$D\beta''(t) = (u'' + (u')^2\Gamma_{11}^1 + 2u'v'\Gamma_{12}^1 + (v')^2\Gamma_{22}^1)x_u$$

$$+ (v'' + (u')^2\Gamma_{11}^2 + 2u'v'\Gamma_{12}^2 + (v')^2\Gamma_{22}^2)x_v,$$

using local coord.

$\Rightarrow$ can extend our def. of geodesics to any Riemannian surface.

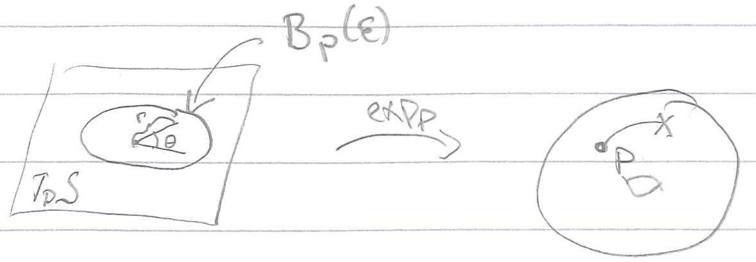
new material

* Geodesic polar coord: $x : \underbrace{B_p(\varepsilon)}_{\subset T_pS} \longrightarrow S$

$$x(r,\theta) = \underbrace{\exp_p(r(\cos\theta, \sin\theta))}$$

$r \mapsto \exp_p(r(\cos\theta, \sin\theta))$ the geodesic satisfying $\exp_p(0) = p$,

$\exp_p'(0) = \underbrace{(\cos\theta, \sin\theta)}_{\in T_pS}$



Metric in these coord:

$$ds^2 = \underbrace{1}_{r \mapsto \exp_p(r(\cos\theta, \sin\theta)) \text{ of arc-length}} \cdot dr^2 + \overset{\text{Gauss' lma}}{0} \cdot dr\,d\theta + G(r,\theta)\,d\theta^2$$

Means: if $v, w \in T_pS$, $v = v_1 X_r + v_2 X_\theta$, $w = w_1 X_r + w_2 X_\theta$

then $\langle v, w\rangle_p = 1 \cdot v_1 w_1 + 0 \cdot (v_1 w_2 + v_2 w_1) + G(r,\theta) v_2 w_2$

__Rmk__ Can prove that $G = h^2(r,\theta)$ where h solves

$$K = -\frac{h_{rr}}{h}.$$ See book p 151-152.

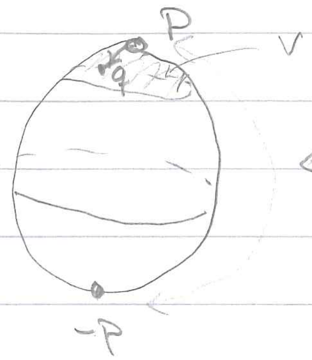
As a consequence we get

__Thm 5.8.1__ Suppose S Riemannian surface w/ constant K. Then

- If $K=0$, S is locally isometric to $\mathbb{E}^2$
- If $K = 1/R^2$, S —"— a sphere of radius R
- If $K = -1/\rho^2$, S —"— the scaled hyperbolic plane $\mathbb{D}_\rho$ $\left(= \mathbb{D} \text{ w/ metric } ds^2 = dr^2 + \rho^2 \sinh^2\frac{r}{\rho}\, d\theta^2 \right.$ See recommended exercises$\left.\right)$

Geodesics locally length-minimizing:

Thm 5.7.6 Every pt p in a Riemannian surface S has a nbhd V s.t. any pt $q \in V \setminus \{p\}$ can be connected to p by a unique shortest curve and this curve is a geodesic.

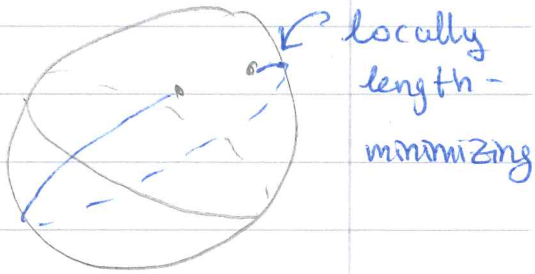


! ← no unique shortest curve connecting p & -p

Pf See book.

Def A curve $\beta : [a,b] \to S$ is locally length-minimizing if any $c \in (a,b)$ is contained in a $(a',b') \subset (a,b)$ s.t.

$$\ell(\beta|_{(a',b')}) = \underbrace{d(\beta(a'),\beta(b'))}_{= \inf\limits_{\substack{\alpha \text{ curve} \\ \text{from } a' \text{ to } b'}} \ell(\alpha)}$$



Cor (= Thm 5.6.14) A constant speed curve is locally length-minimizing $\iff$ it is a geodesic.

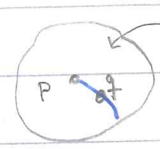
Geodesically complete surfaces

Def S is a geodesically complete surface if $\exp_p$ is defined on all of T_pS $\forall p \in S$.

Non-ex: V a normal nbhd of S, remove a pt $\overset{\not p}{q} \in V$

$\Rightarrow S \setminus \{q\}$ will not be geodesically complete

$V = \exp_p(B_p(\varepsilon))$



if remove q, the blue geodesic has to stop.

Thm 5.6.13 (Hopf-Rinow)

A Riemannian surface is geodesically complete $\Leftrightarrow$ it is complete in the metric d induced by the Riemannian structure. Moreover, if S is complete then any $p, q \in S$ can be joined by a geodesic of length $d(p,q)$.

Cor If S geodesically complete, then $\exp_p : T_pS \to S$ is surj $\forall p \in S$

Cor All cpct Riemannian surfaces are geodesically complete.

Cor $S \subset \mathbb{R}^3$ regular & closed as a subset of $\mathbb{R}^3$ is geodesically complete

Pf $d(p,q) \geq |p-q| \Rightarrow$ A Cauchy seq in (S,d) is a Cauchy seq in $(\mathbb{R}^3, |\cdot|) \Rightarrow$ has a limit pt in S if S closed. $\square$

§5.9 The Gauss-Bonnet thm

A global property of Riemannian surfaces that relates the topology of the surface w/ the Gaussian curvature.

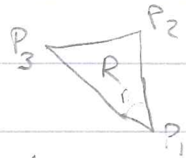
Statement: Thm 5.9.2 (Gauss-Bonnet)

$$\iint_R K\,dA + \int_{\partial R} k_g\,ds + \sum_{i=1}^{n} \varepsilon_i = 2\pi\chi(R)$$

$R \subset S$ cpct region bounded by

∂R = a finite nbr of piecewise smooth, regular, closed curves

w/ $p_1, \dots, p_n$ as non-smooth points



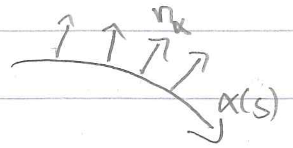
ε_i : Let $\eta_i \in [0, 2\pi]$ be the interior angle of p_i,

R η_i p_i let $\varepsilon_i = \pi - \eta_i \in [-\pi, \pi]$

(the change of dir of ∂R at p_i)

k_g - Geodesic curvature : If $\alpha(s) \subset S$ regular curve param by arc-length, let $n_\alpha(s) \in T_{\alpha(s)}S$ a unit normal vector $(\langle \alpha'(s), n_\alpha(s)\rangle = 0, \|n_\alpha(s)\| = 1 \ \forall s)$.

A cont. choice of $n_\alpha(s)$ is a <u>normal orientation</u> of α.



<u>Def 5.9.1</u> Let $\alpha(s)$ be a normally oriented curve param. by arc length. The <u>geodesic curvature</u> $k_g(s)$ is the normal component of $D\alpha''(s)$:

$$k_g(s) = \langle D\alpha''(s), n_\alpha(s) \rangle$$

<u>Rmk</u> * can prove $D\alpha''(s) = k_g(s) n_\alpha(s)$ $\forall$ surfaces S

(clearly holds for $S \subset \mathbb{R}^3$ regular)

$\Rightarrow$ $\boxed{\alpha(s) \text{ geodesic} \iff k_g(s) = 0 \ \forall s}$

$\Rightarrow$ k_g is a measure of how far the curve is from being a geodesic

* k_g changes sign if we change normal orient.

Choose normal orient of ∂R by requiring the normal to point into R

[defined on each smoot component]

Also need to define:

Line integrals If $f: C \to \mathbb{R}$ fcn, $C \subset S$ curve

param by $\alpha : [a,b] \to S$ s.t. $\alpha'(t) \neq 0 \ \forall t$. Then

$$\int_C f\,ds = \int_a^b f(\alpha(t)) \frac{ds}{dt} dt = \int_a^b f(\alpha(t)) \|\alpha'(t)\|_{\alpha(t)} dt$$

Surface integrals $f: R \to \mathbb{R}$, $R \subset S$ cpct region bdd

by a piecewise smooth curve.

If $R \subset x(U)$, $x: U^{\subset \mathbb{R}^2} \to S$ local param, then define

$$\iint_R f\,dA = \iint_{x^{-1}(R)} f(x(u,v)) \sqrt{EG - F^2}\, du\, dv$$

In the general case, let $R = \bigsqcup_{i=1}^m R_i$ s.t. each

only intersecting along their bdries

R_i is contained in a local param & define

$$\iint_R f\,dA = \sum_{i=1}^m \iint_{R_i} f\,dA$$

Rmk Can prove the integral is indep. of the choice of

local param.

Rmk If S cpct, then $\iint_S f\,dA$ defined $\Rightarrow$ if S closed

$$\iint_S K\,dA = 2\pi \chi(S)$$

$\Rightarrow$

Cor of Gauss-Bonnet:

* For any metric on S^2 & P^2 there must be pts where $K>0$.

* For $\#_n T^2$, $n \geq 2$, there must be pts where $K<0$.

* On T^2 & K^2 (Klein bottle) the curvature either vanishes everywhere, or it must take both pos & neg values.

Q Why?

Cor The only closed surfaces w/ metrics of

K const & pos are S^2 & P^2

$K=0$ T^2 & K^2

And none of these 4 can have metrics of constant negative Gaussian curvature

More consequences:

* If $R \subset S$ w/ angles α, β, γ & $K|_R$ constant, then

$$K \cdot \underbrace{A(R)}_{\text{area of } R} = \alpha+\beta+\gamma-\pi \Rightarrow \alpha+\beta+\gamma=\pi \text{ in } \mathbb{E}^2$$

$$A(R) = \pi - (\alpha+\beta+\gamma) \text{ in } \mathbb{H}/\mathbb{D}$$

$$A(R) = \alpha+\beta+\gamma-\pi \text{ in } S^2/P^2$$

* $\chi(S)$ is indep. of triangulation

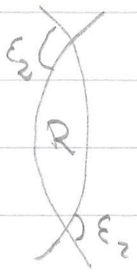
Also some results on how geodesics can behave:

If S has $K \leq 0$ then

* a closed geodesic cannot bound a disk R

$$\left[k_g = 0, \quad \underbrace{\iint_R K\,dA}_{\leq 0} = \underbrace{2\pi - \varepsilon}_{>0}, \quad \varepsilon \leq \pi \right]$$

* 2 geodesics cannot meet at 2 pts to bound a region $R \approx$ disk



$$\left(k_g = 0, \quad \underbrace{\iint_R K\,dA}_{\leq 0} = \underbrace{2\pi - \varepsilon_1 - \varepsilon_2}_{>0} \quad \varepsilon_i < \pi \right)$$

If S closed, orientable w/ $K > 0$, then 2 $\underbrace{\text{simple closed}}_{\approx S^1}$ geodesics must have a pt of intersection

$$\left[K > 0 \Rightarrow \chi(S) > 0 \Rightarrow S = S^2 \right.$$

If the geodesics do not intersect they bound a cylinder $R \approx S^1 \times I$ & $\chi(R) = 0$ but

Gauss-Bonnet $\Rightarrow 0 = 2\pi\chi(R) = \iint_R K\,dA > 0$ ↯

For the pf of the Gauss-Bonnet thm, see

the book (the proof is not a part of the exam