# §2 Hyperbolic geometry

**Last time** Hilbert's axioms

- Incidence
- Betweenness
- Congruence
- Continuity - Axiom E
- Parallels - Axiom P

$\Rightarrow \mathbb{E}^2$

+ $\underline{H}$: Given a line $l$ and a point $P \notin l$, there are at least 2 lines through $P$ which do not intersect $l$.

P: at most

$\rightsquigarrow$ Hyperbolic geometry

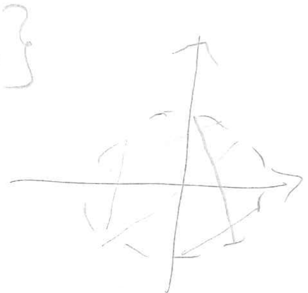
**Short history**: * Can define geometry w/o the parallel axiom: Lobachevski, Bolyai, Gauss indep.

* First concrete model: Beltrami in 1868
* Name "Hyperbolic geometry" introduced by Klein in 1871

$\rightsquigarrow$ The Beltrami-Klein model $\mathbb{K}$ of the hyperbolic plane:

Interior of the unit disk $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$

$\{\text{lines}\} = \{ l \cap D \mid l \text{ line in } \mathbb{R}^2,\ l \cap D \neq \emptyset \}$



**Q** Which of $I^*, B^*, C^*, E$ & $H$ hold if we use the same def. of betweenness & congruence as in $\mathbb{E}^2$

$\Rightarrow$ must find another way to define congruence

To that end, will use other models of $\mathbb{K}$, where congruence can be defined in terms of Möbius transform

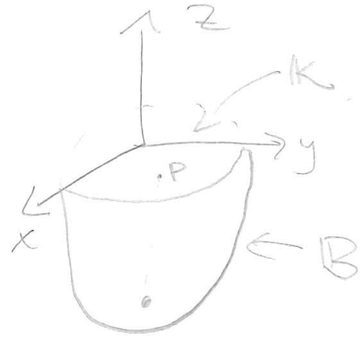
(sim. to $\mathbb{E}^2$: congruence def. in terms of Euclidean gp transformation)

2 different models: $\mathbb{D}$ - the Poincaré disk [set = D, other lines]

$\mathbb{H}$ - the Poincaré upper half-plane

To go from $\mathbb{K}$ to these models, use

## §2.1 Stereographic projection




Let $\mathbb{B} = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1,\ z<0\}$

Define vertical projection

$P_v : \mathbb{K} \longrightarrow \mathbb{B}$

$(x,y) \longmapsto (x, y, -\sqrt{1-x^2-y^2})$ $\qquad P \mapsto$ (line $(x,y)=$const through $P$) $\cap \mathbb{B}$

Chords (lines) $\subset \mathbb{K} \xmapsto{P_v}$ (open) semi-circles in $\mathbb{B}$ meeting $\{(x,y,z) ; x^2+y^2=1, z=0\}$ $\perp$ (*)

$\ell \longmapsto$ (plane containing $\ell$ & vertical line through a point on $\ell$) $\cap \mathbb{B}$

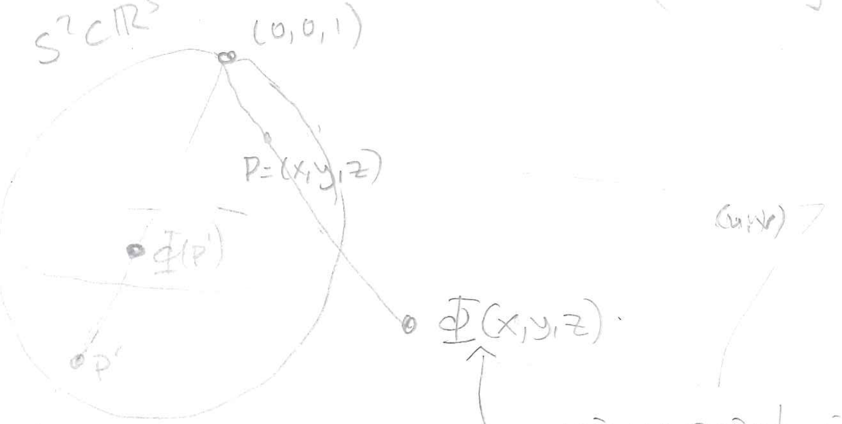

"B-lines"

$\Rightarrow$ $\mathbb{B}$ w/ $\mathbb{B}$-lines gives another model of the hyperbolic plane.

To get $\mathbb{D}$ – use $\boxed{\text{stereographic projection}}$ to map $\mathbb{B}$ back to $\mathbb{D}$

$\rightarrow$ $\Phi : S^2 \setminus (0,0,1) \longrightarrow \mathbb{R}^2$, $\Phi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

w/ inverse $\Phi^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$




unique point in plane $\{z=0\}$ on the line through $(0,0,1)$ & $P$

Q Where is $(0,0,-1)$ mapped?

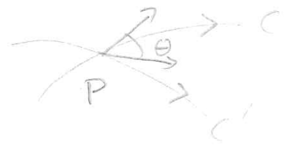
Exc 2.1.1 Derive the formulas for $\Phi$ & $\Phi^{-1}$ (PSS1)

Important properties

Def $C$ a directed / oriented curve: a curve together w/ a choice of nonzero tangent vector at every point

in $\mathbb{R}^2/\mathbb{R}^3/S^2$

Def Let $C, C' \subset \mathbb{R}^2/\mathbb{R}^3$ be 2 oriented curves intersecting in a point $P$. Then the angle between $C$ and $C'$ at $P$ is the angle between their oriented tangent vectors.




That is, if $v$ is a tangent vector of $C$ at $P$, $w$ a tangent vector of $C'$ at $P$, wr correct orientations, then

$$\frac{v \cdot w}{\|v\| \|w\|} = \cos \theta$$

Q Angle between $(1,0,0)$ & $(0,1,0)$? $(1,1,1)$ & $(2,2,2)$? $(-1,1,1)$ & $(1,0,1)$?

Lma 2.1.1 Let $C$ be a circle on $S^2$

(i) If $(0,0,1) \notin C$, then $\Phi(C)$ is a circle in $\mathbb{R}^2$

(ii) If $(0,0,1) \in C$, then $\Phi(C - (0,0,1))$ is a straight line in $\mathbb{R}^2$.

Lma 2.1.2 $\Phi$ preserves angles:

If $C, C'$ oriented circles $\subset S^2$ intersecting in $P$ at an angle $\theta$, then $\Phi(C)$ & $\Phi(C')$ intersect in $\Phi(P)$ at the same angle

Will prove this in 2nd half. For now, use this to define

1. <u>The Poincaré disk model $\mathbb{D}$</u> $= \Phi(\mathbb{B})$ w/ $\mathbb{B}$ as before 

$\Rightarrow \mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$

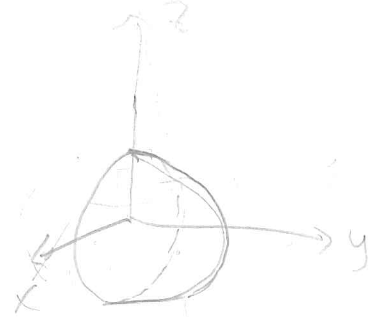
lines = <u>$\mathbb{D}$-lines</u> = $\begin{cases} \text{• diameters} \\ \text{• circular arcs } \perp \text{ to the boundary circle} \end{cases}$

<u>Q</u> Why is this?

2. <u>The Poincaré upper half plane</u>

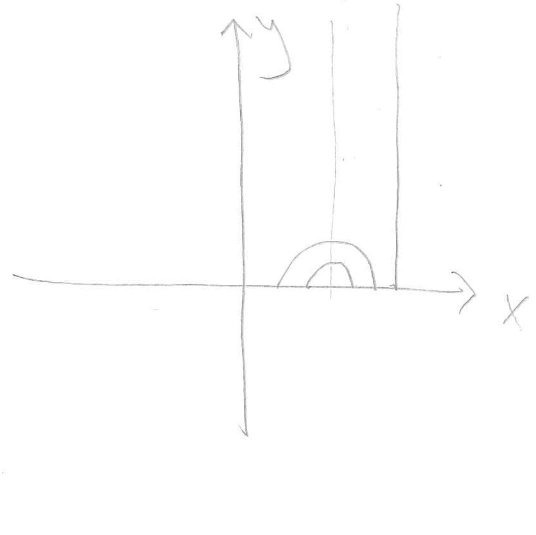
Now let $\mathbb{B}' = \{(x,y,z) \in S^2 \mid y>0\}$

$\mathbb{H} = \Phi(\mathbb{B}') = \{(x,y) \in \mathbb{R}^2 \mid y>0\}$




lines = $\mathbb{H}$-lines = $\begin{cases} \text{• semi-circles w/ center on x-axis} \\ \text{• straight lines } \parallel \text{ to y-axis} \end{cases}$

<u>Q</u> Why?



From now on — Identify $\mathbb{R}^2 = \mathbb{C}$ $\Rightarrow$ get more tools!

$$\Rightarrow \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$$

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

Important models in many fields of mathematics!

Notation — Endpoint of a hyperbolic line ($= \mathbb{D}$ or $\mathbb{H}$-line)

= a point in the extension of the line to

$\partial \mathbb{D}$ / $\mathbb{R}$-axis


⚠ These are not points on the lines!

Pf of Lma 2.1.1

Circle $C$ on $S^2 = S^2 \cap \underbrace{\{ax + by + cz = d\}}_{\text{plane}}$

$(0,0,1) \in C \Leftrightarrow c = d$

Assume $(x,y,z) \in C$ & let $(u,v) = \Phi(x,y,z)$ $\Rightarrow$

$$(x,y,z) = \Phi^{-1}(u,v) = \left( \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$$

$$\Rightarrow \frac{2au}{u^2+v^2+1} + \frac{2bv}{u^2+v^2+1} + \frac{c(u^2+v^2-1)}{u^2+v^2+1} = d$$

in eqn for the plane

$\Leftrightarrow 2au + 2bv + c(u^2+v^2) - c = d + d(u^2+v^2)$

$\Leftrightarrow (c-d)(u^2+v^2) + 2au + 2bv = c + d$

If $c = d$, this is the eqn of a line

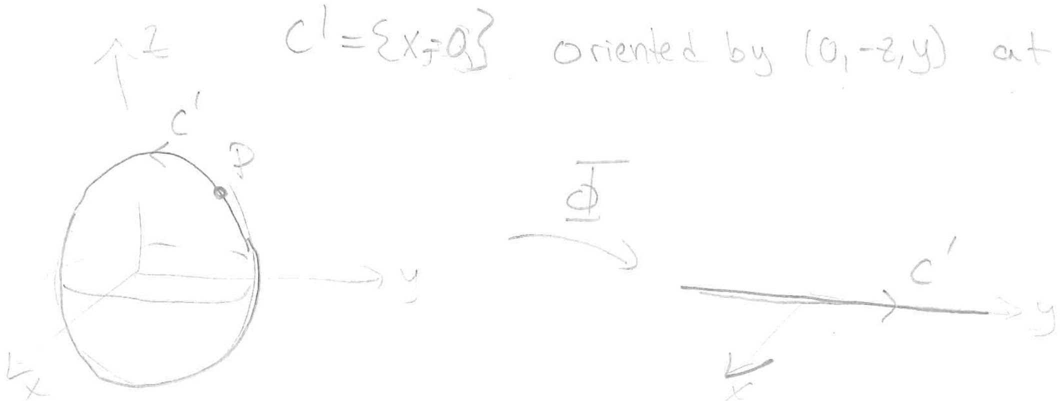
If $c \neq d$, this is the eqn of a circle $\left( \begin{array}{l} u^2 - \tilde{a}u + v^2 - \tilde{b}v = \tilde{c} \\ (u - \frac{\tilde{a}}{2})^2 + (v - \frac{\tilde{b}}{2})^2 = \tilde{c} + \frac{\tilde{a}^2 + \tilde{b}^2}{4} \end{array} \right)$

$\square$

## Pf of Lma 2.1.2

W.l.o.g $P = (0, \overset{>0}{y}, z)$ (rotational symmetry)

$C' = \{x = 0\}$ oriented by $(0, -z, y)$ at $(0, y, z)$




$\Phi(C') = y$-axis

Oriented by $(0,1)$

Chain rule $\Rightarrow$ tangential curves map to tangential curves.

Q Show this

[ $\gamma_1, \gamma_2$ curves w/ $\gamma_1(0) = \gamma_2(0)$, $\gamma_1'(0) = \gamma_2'(0)$ $\Rightarrow$ $\Phi(\gamma_1(0)) = \Phi(\gamma_2(0))$

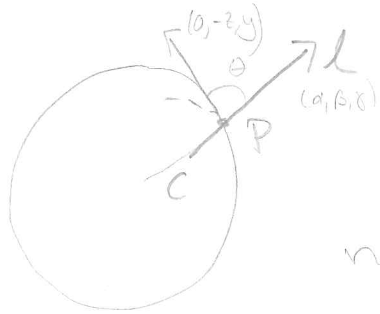
$\Phi'(\gamma_1(0)) = \Phi'(\gamma_2(0))$

Pf $\Phi(x,y,z) = (\Phi_u(x,y,z), \Phi_v(x,y,z))$ $\quad \Phi'(\gamma_i(0)) = d\Phi(\gamma_i(0)) \cdot \gamma_i'(0)$ [chain rule]

$$d\Phi(P) = \begin{bmatrix} \frac{\partial \Phi_u}{\partial x}(P) & \frac{\partial \Phi_u}{\partial y}(P) & \frac{\partial \Phi_u}{\partial z}(P) \\ \frac{\partial \Phi_v}{\partial x}(P) & \frac{\partial \Phi_v}{\partial y}(P) & \frac{\partial \Phi_v}{\partial z}(P) \end{bmatrix} \qquad \gamma_i'(0) = (\gamma_{i,x}'(0), \gamma_{i,y}'(0), \gamma_{i,z}'(0))$$

+ can extend $\Phi$ to $\{(x,y,z) \in \mathbb{R}^3 \mid z<1\}$ by the same formula

$\Rightarrow$ can replace $C$ by a straight line $\subset \mathbb{R}^3$ tangent to $C$ at $P$




$$\ell(t) = (0,y,z) + t(\alpha,\beta,\gamma)$$

normalized so that $\alpha^2+\beta^2+\gamma^2=1$

<u>Recall</u> Angle $\Theta$ between $C$ & $C'$ at $P$ given by

(*) $$\underbrace{(\alpha,\beta,\gamma)}_{\|\ \|=1} \cdot \underbrace{(0,-z,y)}_{\|\ \|=1} = \cos\Theta$$

<u>Remains</u> calculate angle $\Theta'$ between $\Phi(\ell(t))$ & $\Phi(C') = y$-axis

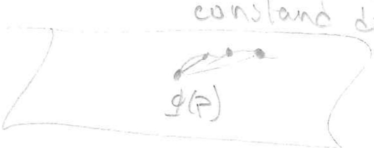
<u>Claim 1</u> $\Phi(\ell(t))$ is a straight line w. direction

$$V = \left(\alpha, \frac{\beta - \beta z + \gamma y}{1-z}\right)$$

<u>Pf</u> $\Phi(\ell(t)) = \left(\frac{t\alpha}{1-z-t\gamma}, \frac{y+t\beta}{1-z-t\gamma}\right)$, enough to consider part where $\underbrace{1-z}_{>0}-t\gamma > 0$

[keep $|t|$ small]

$$\Rightarrow \underbrace{\Phi(\ell(t)) - \Phi(P)}_{\text{want this to have constant direction}} = \left(\frac{t\alpha}{1-z-t\gamma}, \frac{y+t\beta}{1-z-t\gamma} - \frac{y}{1-z}\right)$$




$$= \frac{t}{1-z-t\gamma} \underbrace{\left(\alpha, \frac{\beta - z\beta + \gamma y}{1-z}\right)}_{V}$$

$\Delta$ Claim

Let $V = (V_1, V_2) \Rightarrow V_2 = \frac{\beta - z\beta + \gamma y}{1-z}$

$\Rightarrow$ Remains to compute $\cos\Theta' = (0,1)\cdot \frac{V}{\|V\|} = \frac{V_2}{\|V\|}$

Claim 2 $\quad V_2 = \cos\theta \quad , \|V\| = 1$

$\Rightarrow$ Lemma will follow from claim 2.

Pf of Claim 2

$z\cos\theta = \left[ (*) \Rightarrow \cos\theta = -\beta z + \gamma y \right] = -\beta z^2 + \gamma z y = \left[ \begin{array}{l} \beta y + \gamma z = 0 \\ \text{since} \\ (\alpha, \beta, \gamma) \text{ tangent} \\ \text{to } S^2 \text{ at} \\ (0, y, z) \end{array} \right]$

$= -\beta z^2 - \beta y^2 = \left[ z^2 + y^2 = 1 \text{ since } (0,y,z) \in S^2 \right] = -\beta$

$y\cos\theta = -\beta z y + \gamma y^2 = \left[ \text{similarly} \right] = \gamma z^2 + \gamma y^2 = \gamma$

$\Rightarrow V_2 = \dfrac{\beta - \beta z + \gamma y}{1 - z} = \dfrac{-z\cos\theta + \cos\theta}{1 - z} = \cos\theta \quad \checkmark$

$\Rightarrow V = (\alpha, \cos\theta)$

Also, $\beta^2 + \gamma^2 = z^2\cos^2\theta + y^2\cos^2\theta = \cos^2\theta$

$\Rightarrow \|V\| = \alpha^2 + \cos^2\theta = \alpha^2 + \beta^2 + \gamma^2 = 1. \quad \checkmark$

$\triangle$ Claim

Hence the lemma is proven $\square$

Geometric interpret of this: See book.