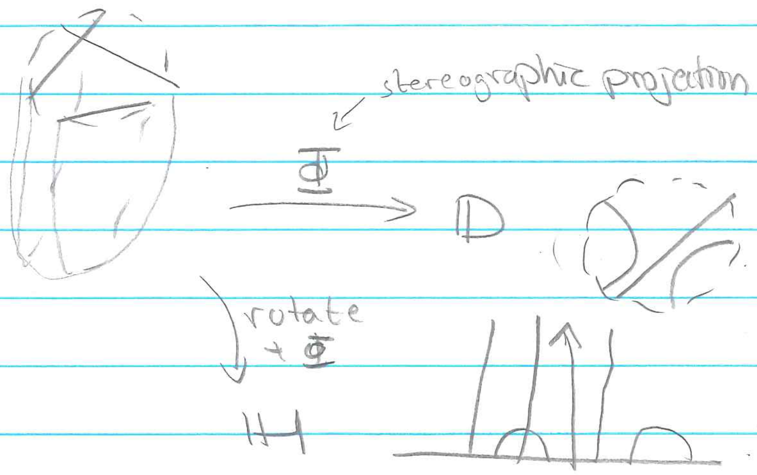
Last time: Different models for the hyperbolic plane:

$\mathbb{K}$

$\mathbb{B}$



<u>Problem</u> How to define congruence?

§ 2.2 Congruence in $\mathbb{H}$: Möbius transformations

In $\mathbb{E}^2$: Segments: $AB \cong A'B' \iff \exists g \in E(2)$ s.t. $g(AB) = A'B'$

Angles: $\angle BAC \cong B'A'C' \iff \exists g \in E(2)$ s.t.

$$g(\overrightarrow{AB}) = \overrightarrow{A'B'} \;\&\; g(\overrightarrow{AC}) = \overrightarrow{A'C'}$$

$E(2)$ = the Euclidean gp $= \{ g: \mathbb{R}^2 \to \mathbb{R}^2,\ g$ generated by $A \in O(2)$ & $x \mapsto x+b \}$.

<u>Rmk</u> * If $g \in E(2)$ & l line, then $g(l)$ line

* If $A \in l$, $A' \in l'$, then $\exists g \in E(2)$ s.t. $g(l) = l'$, $g(A) = A'$

Mimic this for $\mathbb{H}$ $\Rightarrow$

find gp G of bijections $\mathbb{H} \circlearrowleft$:

* $g \in G$ & l a $\mathbb{H}$-line $\Rightarrow$ $g(l)$ a $\mathbb{H}$-line
* if l, l' $\mathbb{H}$-lines, $A \in l$, $A' \in l'$ then $\exists\, g \in G$ s.t.

  $g(l) = l'$, $g(A) = A'$.

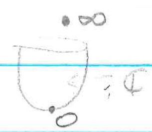
Today: Show existence of G

$= \{ \underbrace{\text{Möbius transformations}}_{1}$ $\underbrace{\text{preserving the upper half plane}}_{2} \}$.

Riemannian sphere

Notation $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ w/ topology so that

$\overline{\Phi} : S^2 \to \overline{\mathbb{C}}$ homeo when added ∞



The Riemann sphere

$\overline{\mathbb{C}}$-circles $= \begin{cases} * \text{ circles in } \mathbb{C} \\ * \; l \cup \{\infty\}, \; l \text{ a (real, 1-dim) line in } \mathbb{C} \end{cases}$

Last time: Circles in $S^2 \xleftrightarrow{\overline{\Phi}} \overline{\mathbb{C}}$-circles

$\overline{\mathbb{R}} = \mathbb{R} \cup \infty \subset \overline{\mathbb{C}}$

Recall $f: \mathbb{C} \to \mathbb{C}$ analytic (holomorphic) if complex diffable at each pt $\iff$

$f(z) = f_u(x+iy) + i f_v(x+iy)$, $\frac{\partial f_u}{\partial x} = \frac{\partial f_v}{\partial y}$, $\frac{\partial f_u}{\partial y} = -\frac{\partial f_v}{\partial x}$

$\boxed{\text{Cauchy-Riemann eqn}}$

$f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ meromorphic if analytic except at isolated points which are poles ($f(\text{pole}) = \infty$)

$\Rightarrow f = \frac{f_1}{f_2}$, f_1, f_2 analytic, poles of f = zeros of f_2

Q Examples of analytic fcns

FLT:S

Notice If a meromorphic $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism then it has exactly 1 pole & 1 zero

$\Rightarrow f(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$.

First consider this type of fcn in generality, then see what it has to satisfy to restrict to a map $f: \mathbb{H} \to \mathbb{H}$.

Lma A $f(z) = \frac{az+b}{cz+d}$ is a homeo $f: \overline{\mathbb{C}} \to \overline{\mathbb{C}} \iff$ $ad - bc \neq 0$.

Pf Let $f(z) = w$ $\Rightarrow$ must find $g: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ s.t. $g(f(z)) = z$
$f(g(w)) = w$.

Solve $w = \frac{az+b}{cz+d}$ for z:

$$w(cz+d) = az+b \Leftrightarrow z = \frac{b-wd}{cw-a} \; (= g(w))$$

$$\Rightarrow f(g(w)) = \left[\frac{a\frac{b-wd}{cw-a} + b}{c\frac{b-wd}{cw-a} + d} \right] = \frac{ab - awd + bcw - ba}{cb - cwd + dcw - ad}$$

$$= \frac{(ad-bc)w}{ad-bc}$$

$$g(f(z)) = [\text{similarly}] = \frac{(ad-bc)}{ad-bc} z$$

$\Rightarrow$ If $ad-bc \neq 0$, this makes sense & $\boxed{g(w) = \frac{-wd+b}{cw-a}}$

gives an inverse to f. $\square$

Def A function $f = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ is called a

Fractional Linear Transformation (FLT).

Properties of FLT's

Lma / Def B The set of FLT's forms a gp under composition

$Mōb^+(\mathbb{C})$ = "the even complex Möbius transformations"

Pf Inverse already checked

Identity $a = d = 1$, $b = c = 0$

Composition : $f(z) = \frac{az+b}{cz+d}$, $g(z) = \frac{a'z+b'}{c'z+d'}$

$$\Rightarrow (f \circ g)(z) = \frac{\overbrace{(aa'+bc')}^{\tilde{a}}z + \overbrace{(ab'+bd')}^{\tilde{b}}}{\underbrace{(ca'+dc')}_{\tilde{c}}z + \underbrace{(cb'+dd')}_{\tilde{d}}}$$

$\Rightarrow$ this is an FLT if $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} \neq 0$

Notice $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{bmatrix}}_{C}$

$\Rightarrow$ can calculate w/ FLT's as w/ matrices!

$\det A = ad - bc$, $\det B = a'd' - c'b'$

$\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = \det C = \underbrace{\det A}_{\neq 0} \cdot \underbrace{\det B}_{\neq 0}$ $\Rightarrow$ $f \circ g$ is an FLT $\square$

$\Rightarrow \text{Möb}^+(\mathbb{C}) \simeq PGL_2(\mathbb{C}) = GL_2(\mathbb{C}) / \{uI \mid u \in \mathbb{C} \setminus \{0\}\}$

the projective linear group

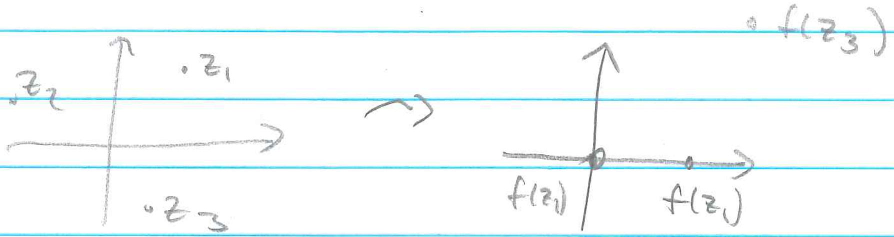
invertible 2x2-matrices / $\mathbb{C}$

use I & kI mapped to the same FLT

FLT's are nice, because they are easy to describe explicitly:

<u>Lma 2.2.3</u> Given 3 distinct points $z_1, z_2, z_3 \in \bar{\mathbb{C}}$.

Then $\exists!$ FLT f s.t. $f(z_1) = 1$, $f(z_2) = 0$, $f(z_3) = \infty$



<u>Pf</u> <u>Existence</u> If none of the z_i's is at ∞:

$$f(z) = \frac{z - z_2}{z - z_3} \frac{(z_1 - z_3)}{(z_1 - z_2)}$$

<u>Q</u> In the other cases? $z_1 = \infty$ $\frac{z - z_2}{z - z_3}$

$z_2 = \infty$ $\frac{z_1 - z_3}{z - z_3}$

$z_3 = \infty$ $\frac{z - z_2}{z_1 - z_2}$

<u>Uniqueness</u> Suppose $g(z)$ also satisfies this, consider

$h = g \circ f^{-1}$. Lma B $\Rightarrow$ this is an FLT &

$h(1) = 1$, $h(0) = 0$, $h(\infty) = \infty$ $\Rightarrow h(z) = az + b$ ["no "true" poles"]

$\Downarrow$ $a = 1$ $\Downarrow$ $b = 0$

$\Rightarrow (g \circ f^{-1})(z) = z \; \forall z \Rightarrow f = g$ □

Möb$^+(\mathbb{C})$ acts transitively

M

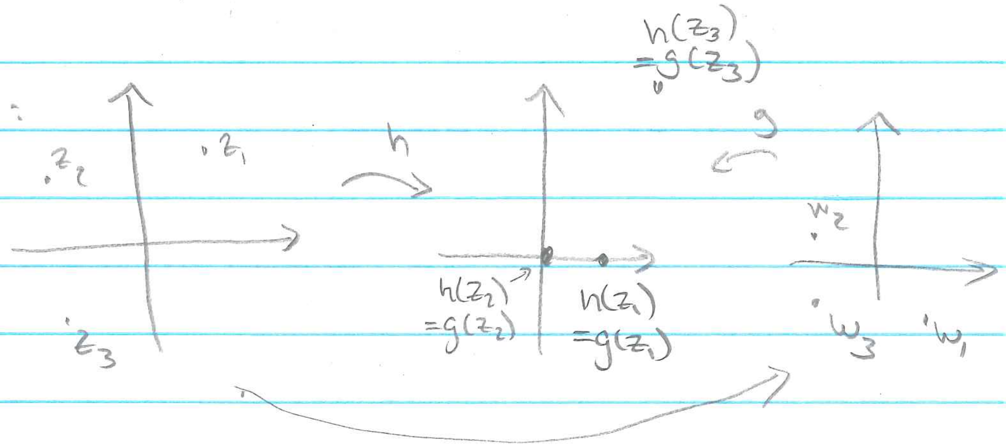
Cor 2.2.5 Möb$^+(\mathbb{C})$ acts transitively on the set of triples of distinct points in $\mathbb{C}$:

(given 2 such triples (z_1, z_2, z_3) & (w_1, w_2, w_3) $\exists$ an FLT f s.t. $f(z_i) = w_i$, $i = 1,2,3$)

In fact, f is uniquely determined, and if all 6 points $\in \mathbb{R}$, then $f(z) = \dfrac{az+b}{cz+d}$ w/ $a,b,c,d \in \mathbb{R}$.

Pf. Follows from Lma 2.2.3.

Existence of f:



$f = g^{-1} \circ h$.

Uniqueness Suppose also $f' \in$ Möb$^+(\mathbb{C})$ maps z_i to w_i, $i=1,2,3$

$\Rightarrow gf = gf'$ by uniqueness in Lma 2.2.3 $\Rightarrow f = f'$.

Real coeff: Follows from the formula in Lma 2.2.3 + formula for inverse of FLT's.

Q Check this.

$\square$

Now recall what we want to do:

Find a gp of bijections of $\mathbb{H}$ mapping $\mathbb{H}$-lines to $\mathbb{H}$-lines

w/ point constraints.

Recall $\mathbb{H}$-line: either * line $\parallel$ y-axis $\subset \{y>0\}$

* semi-circle w/ center on x-axis

Note Both these can be extended to a $\overline{\mathbb{C}}$-circle.

We are also interested in maps preserving angles:

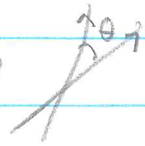
Note If C, C' are "true" circles intersecting. Then if

$C \cap C' = \{P_1, P_2\}$, the angles of intersections coincide

$C \cap C' = \{P\}$ —— " —— $= 0$ (tangent)

Define angle between 2 $\overline{\mathbb{C}}$-circles intersecting at ∞ as:

* 0 if only pt of intersection 

* θ if intersect at angle θ at another point P 

<u>Lma 2.2.1</u> (i) An FLT maps $\bar{\mathbb{C}}$-circles to $\bar{\mathbb{C}}$-circles

ii) An FLT preserves angles between $\bar{\mathbb{C}}$-circles

<u>Pf</u> i) Sim to pf of Lma 2.1.1

ii) Follows from the <u>fact</u> that analytic maps are <u>conformal</u> at each pt where their derivative is $\neq 0$

<u>Def</u> A <u>conformal</u> map is a map that preserves angles & orientations.

Pf of <u>fact</u>: $f : \underset{(x,y)}{\mathbb{R}^2} \longrightarrow \underset{(u,v)}{\mathbb{R}^2}$ conformal if

$$df = \begin{bmatrix} \frac{\partial f_u}{\partial x} & \frac{\partial f_u}{\partial y} \\ \frac{\partial f_v}{\partial x} & \frac{\partial f_v}{\partial y} \end{bmatrix}$$ has pos. determinant & $\perp$ columns.

rotation matrix times scaling

If f analytic w/ $df \neq 0 \Rightarrow df = \begin{bmatrix} \frac{\partial f_u}{\partial x} & -\frac{\partial f_v}{\partial x} \\ \frac{\partial f_v}{\partial x} & \frac{\partial f_u}{\partial y} \end{bmatrix}$ [C-R eqns]

has det $\left(\frac{\partial f_u}{\partial x}\right)^2 + \left(\frac{\partial f_v}{\partial x}\right)^2 > 0$ & $\perp$ columns. $\square$

Cor 2.2.7 Given 2 $\bar{\mathbb{C}}$-circles C_1 & C_2. Then $\exists$

a FLT f s.t. $f(C_1) = C_2$.

Pf

Q : 3 points $(z_1, z_2, z_3) \in \bar{\mathbb{C}}$ determine a unique $\mathbb{C}$-circle

eqn for circle $x^2+y^2+2ax+2by=c$

Given Q, the result follows from Cor 2.2.5 & Lma 2.2.1.

$\square$

To define congruence,

$\Rightarrow$ we are interested in FLT's which maps $\mathbb{H}$ to $\mathbb{H}$.

Def $M\ddot{o}b^+(\mathbb{H}) = \{ f \in M\ddot{o}b(\mathbb{C}^+) \mid f \text{ preserves } \mathbb{H}) \}$. Cor 2.2.7 $\Rightarrow$ such f maps $\mathbb{H}$-lines to $\mathbb{H}$-lines

Prp 2.2.8 An FLT f restricts to a homeo of $\mathbb{H}$ $\Longleftrightarrow$

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1.$$

($\Rightarrow$ $M\ddot{o}b^+(\mathbb{H})$ subgp of $M\ddot{o}b^+(\mathbb{C})$)

Pf Assume $f(z) = \frac{az+b}{cz+d}$ FLT which preserves $\mathbb{H}$ $\Rightarrow$

$$f(\bar{\mathbb{R}}) = \bar{\mathbb{R}} \overset{\text{Cor 2.2.5}}{\Longrightarrow} a, b, c, d \in \mathbb{R}$$

Also, $a, b, c, d \in \mathbb{R} \Rightarrow f(\bar{\mathbb{R}}) = \bar{\mathbb{R}}$ clear.

In addition f preserves $\mathbb{H}$ $\Longleftrightarrow$ ($\operatorname{Im} f(z) > 0$ if $\operatorname{Im} z > 0$.)

We calculate:

FLT's preserving $\mathbb{H}$.

$$f(z) = \frac{(az+b)}{(cz+d)}\frac{(c\bar{z}+d)}{(c\bar{z}+d)} = \frac{ac|z|^2+bd+(ad-bc)\operatorname{Re}z}{|cz+d|^2} + i\frac{(ad-bc)\operatorname{Im}z}{|cz+d|^2}$$

$\Rightarrow$ If $\operatorname{Im} z > 0$ then $\operatorname{Im} f(z) > 0 \iff ad-bc>0$.

$\Rightarrow$ if we mult a, b, c, d by $\frac{1}{\sqrt{ad-bc}}$, then f is as in the statement. $\square$

<u>Rmk</u> In terms of matrices:

$$\text{Möb}^+(\mathbb{H}) \simeq PSL_2(\mathbb{R}) = \underbrace{SL_2(\mathbb{R})}_{\text{gp of } 2\times2\text{-matrices}/\mathbb{R} \text{ w/ det}=1} / \pm I.$$

⚠ $\text{Möb}^+(\mathbb{C})$ does not contain all circle-preserving homeos of $\overline{\mathbb{C}}$.

<u>Q</u> Give example of one such not in $\text{Möb}^+(\mathbb{C})$.

<u>An answer</u> $\bar{z}$ (complex conjugation) $(\overline{\infty} = \infty)$

Not analytic!

Def $\text{Möb}(\mathbb{C})$ = the group of complex möbius transformations

= the group of homeos of $\bar{\mathbb{C}}$ generated by FLT's & $z \mapsto \bar{z}$,

Let $\text{Möb}(\mathbb{H}) = \{g \in \text{Möb}(\mathbb{C}) \mid g \text{ maps } \mathbb{H} \text{ to } \mathbb{H}\}$,

the real Möbius transformations.

Prp 2.2.9 (1) Every $f \in \text{Möb}(\mathbb{C})$ can be written on exactly one of the forms

$$(*)\ f(z) = \frac{az+b}{cz+d} \quad \text{or } (**)\ f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}, \quad a,b,c,d \in \mathbb{C}, \ ad-bc=1$$

(2) Every $f \in \text{Möb}(\mathbb{H})$ can be written either as

i) $f(z) = \dfrac{az+b}{cz+d}$ $\quad a,b,c,d \in \mathbb{R}, \ ad-bc=1$

ii) $f(z) = \dfrac{a\bar{z}+b}{c\bar{z}+d}$ $\quad$ —||—, $ad-bc=-1$

Q Check $z \mapsto -\bar{z}$ preserves $\mathbb{H}$

$\text{Möb}(\mathbb{H})$ gen. by $\text{Möb}^+(\mathbb{H})$ & $z \mapsto -\bar{z}$. Let $\text{Möb}^-(\mathbb{H}) = \{f \in \text{Möb}(\mathbb{H}) \text{ f of type 2}\}$

Q $\text{Möb}^-(\mathbb{H})$ not a subgp.

The lectures after PSS1:

Show Möb($\mathbb{H}$) is the "correct" gp for defining congruence in $\mathbb{H}$

Pf of Prp 2.2.9

(1) Let $S = \{ f \in \text{Möb}(\mathbb{C}) \mid f$ on form $(*)$ or $(**) \}$

$\Rightarrow \text{Möb}^+(\mathbb{C}) \cup \{ z \mapsto \bar{z} \} \subset S$

Check S closed under composition & inverses

either i) or ii) : $(**)$ not analytic but $(*)$ is.

$ad - bc = 1$: Divide numerator & denominator by a sqr. root of $ad-bc$ ($\mathbb{C} - \{0\}$ connected)

(2) Know $f(z)$ can be written as $(*)$ or $(**)$

$(*)$: then follows from Prp 2.2.8

$(**)$: $\Rightarrow g(z) = -\overline{f(z)}$ gives expression as $(*)$ & since $z \mapsto -\bar{z}$ preserves $\mathbb{H}$ have $g(z)$ preserves $\mathbb{H}$ $\Rightarrow$ $g(z)$ written as in i) $\Rightarrow f(z) = -\overline{g(z)} = \frac{-a\bar{z}-b}{-c\bar{z}-d}$ written as in (ii). □