§2.3 Classification of real Möbius transformations

Möb(ℍ) is so important that we will spend some time classifying the elements, in terms of fixed pts ($z = f(z)$).
(corresponds to eigenvectors of corresponding matrices Exc 2.7.12)

Start w/ Möb$^+$(ℍ).

Lma A If $f \in$ Möb$^+$(ℍ), $f \neq id$, then either

* f has exactly one fixpoint z_0, and $z_0 \in \overline{\mathbb{R}}$

or

* f has 2 fixpoints z_0, z_1, both in $\overline{\mathbb{R}}$

* f has 2 fixpoints in $\mathbb{C}$.

Pf z fixpt of $f \iff z = \frac{az+b}{cz+d}$.

i) $c = 0$: Then $f(\infty) = \infty \Rightarrow \infty$ fixpoint.

We also get $ad = 1 \Rightarrow \frac{1}{d} = a \Rightarrow f(z) = a^2 z + ab$

$\Rightarrow$ fixpt z satisfies $z = a^2 z + ab \iff z(1-a^2) = ab$

If $a^2 = 1$ then $b \neq 0$ (otherwise $f(z) = z$) $\Rightarrow$ only one fixpt.

If $a^2 \neq 1$ get $z = \frac{ab}{1-a^2} \in \mathbb{R}$ as second fixpt.

$\Rightarrow$ if $c=0$ we have either 1 or 2 fixpts, belonging to $\overline{\mathbb{R}}$.

$c \neq 0$: $z = \frac{az+b}{cz+d} \Leftrightarrow cz^2 - (a-d)z - b = 0$

$$\Leftrightarrow z = \frac{a-d \pm \sqrt{(a-d)^2+4bc}}{2c} \underset{ad-bc=1}{=} \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c}$$

$\Rightarrow$ get 3 different cases:

- $(a+d)^2 = 4$: exactly 1 root (in $\mathbb{R}$)
- $(a+d)^2 > 4$: 2 real roots
- $(a+d)^2 < 4$: 2 complex roots. $\square$

@ $c=0$, $a^2 = 1 \Rightarrow (a+d)^2 = 4$, $a^2 \neq 1 \Rightarrow \underset{(a-d)^2+4ad}{\overset{\shortparallel}{(a+d)^2}} > 4$

Let $\tau(f) = (a+d)^2$.

Lma B $\tau(f)$ is invariant under conj. by elements in Möb($\mathbb{H}$).

Pf @ $\mathbb{C}$: The trace of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $= a+d$ is invariant under conjugation by elements of $GL_2(\mathbb{C})$.

$tr(AB) = tr(BA) \Rightarrow tr(B^{-1}AB) = tr(ABB^{-1}) = a+d$

τ also inv. under $*$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto -\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$*$ conj by $-\overline{z}$: $-\overline{\left(\frac{-\overline{z}a+b}{-\overline{z}c+d}\right)} = \frac{az-b}{-cz+d}$ $\square$

This makes it possible to describe the 3 different cases even more explicit:

<u>Prp 2.3.2/def</u> Suppose $f(z) = \frac{az+b}{cz+d} \in \text{Möb}^+(\mathbb{H})$, $f \neq id$.

then f is conjugate (by elements in $\text{Möb}^+(\mathbb{H})$) to

* $z \mapsto z+1$ or $z \mapsto z-1$ if $\tau(f) = 4$
  "Parabolic type"
* exactly one $z \mapsto \eta z$, $\eta \in \mathbb{R}_{>1}$, if $\tau(f) > 4$
  "Hyperbolic type"
* a unique $z \mapsto g_\theta(z) = \dfrac{\cos(\theta)z + \sin(\theta)}{-\sin(\theta)z + \cos\theta}$, $\theta \in (0, \pi)$,
  "Elliptic type" if $\tau(f) < 4$

<u>Def</u> $f \in \text{Möb}^+(\mathbb{H})$ is given on <u>normal form</u> if written as hgh^{-1}, $h \in \text{Möb}^+(\mathbb{H})$, g one of the forms in the prop.

<u>Proof of Prp</u>

<u>Case (1), $\tau(f) = 4$</u> (Parabolic) If $f(\infty) = \infty$ then $c = 0$ & $f(z) = z + \beta$ after renormalization.

If $f(q) = q$ w/ $q \in \mathbb{R}$, let $h(z) = \frac{-1}{z-q} \in \text{Möb}^+(\mathbb{H})$

$\Rightarrow h(q) = \infty$ & $g = h \circ f \circ h^{-1} \in \text{Möb}^+(\mathbb{H})$

w/ $g(\infty) = \infty \Rightarrow g(z) = z + \beta$ for some $\beta \in \mathbb{R}$.

Moreover, since $\beta \neq 0$ we have $k(z) = \frac{z}{|\beta|} \in \text{Möb}^+(\mathbb{H})$

(after normalization) and

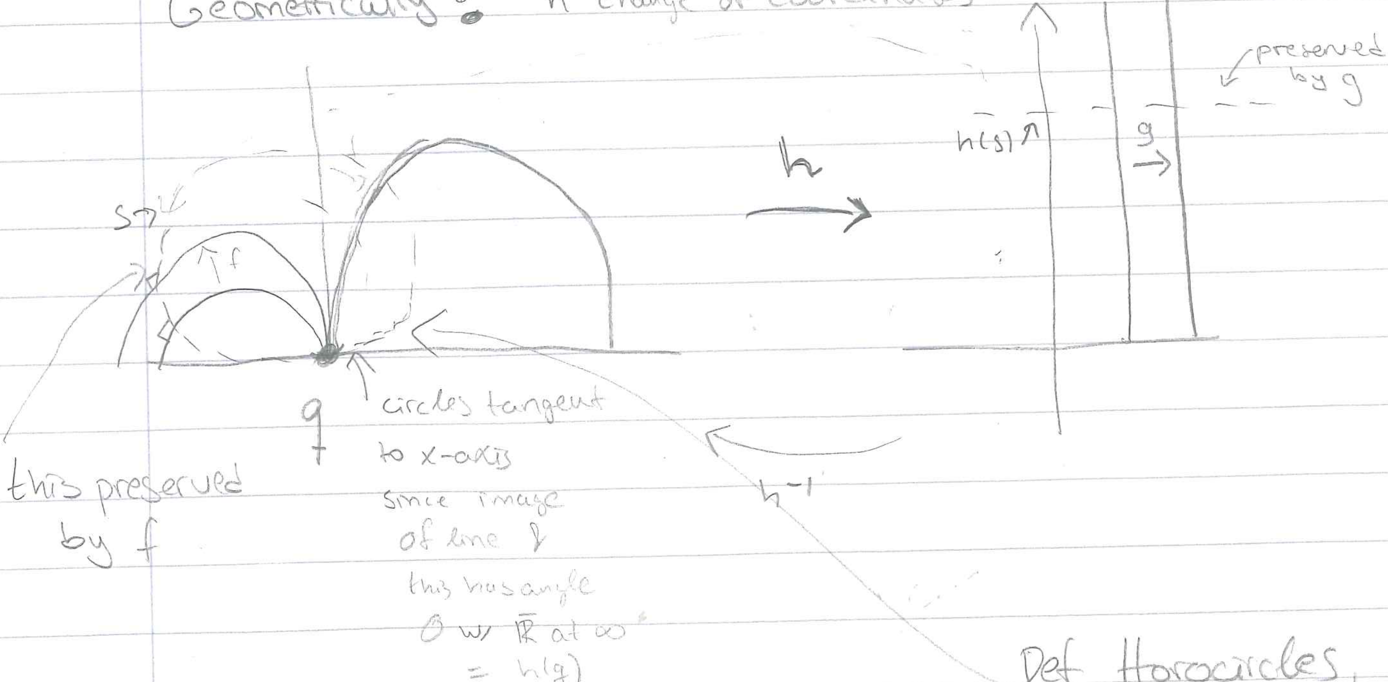
$k \circ g \circ k^{-1}(z) = z \pm 1$.

f if $c = 0$.

Q Check this.

This proves case 1.

Geometrically: h "change of coordinates"



Def Horocircles, tangent to $\mathbb{R}$ at q, intersects all $\mathbb{H}$-lines ending at q $\perp$.

Case (2) $t(f) > 4$, f has 2 fixpts in $\overline{\mathbb{R}}$, p_1, p_2
(Hyperbolic)

Let $h \in \text{Möb}^+(\mathbb{H})$ s.t. $h(p_1) = 0$, $h(p_2) = \infty$

$\Rightarrow g = h \circ f \circ h^{-1} \in \text{Möb}^+(\mathbb{H})$ satisfies $g(0) = 0$
$g(\infty) = \infty$

Q g preserves $i\mathbb{R}_{>0}$ (imaginary axis.) ($i\mathbb{R} \leftarrow \mathbb{H}$-line through 0 & ∞ $= i\mathbb{R}$)

$\Rightarrow \exists\, \eta \in \mathbb{R}_{>0}$ s.t. $g(i) = \eta i$

$\Rightarrow g(z) = \eta z \;\; \forall z \in \mathbb{C}$ since g uniquely determined by the values of 3 pts.

$\Rightarrow f$ conjugate to $g(z) = \eta z$.

Remains: • If $\eta < 1$ then ηz conjugate to $\frac{1}{\eta} z$

• If $\eta_1 z$ and $\eta_2 z$ are conjugate w/ $\eta_1, \eta_2 \in \mathbb{R}_{>0}$, then $\eta_1 = \eta_2$ or $\eta_1 = \frac{1}{\eta_2}$

ηz conjugate to $\frac{1}{\eta} z$:

Q $k = \underbrace{-\frac{1}{z}}_{\substack{a=0 \;\; b=-1 \\ c=1 \;\; d=0 \quad ad-bc=1}}$ satisfies $k^{-1} = k$ & $k \circ \eta z \circ k = \frac{1}{\eta} z$ & $k \in \text{Möb}^+(\mathbb{H})$

Next, if $g_1(z) = \eta_1 z$ & $g_2(z) = \eta_2 z$ are conjugate

then $\tau(g_1) = \tau(g_2)$; Let $\lambda_i > 0$ s.t. $\lambda_i^2 = \eta_i$ $\Rightarrow$

$g_i(z) = \dfrac{\lambda_i z}{\frac{1}{\lambda_i}}$ repr by matrix $\begin{bmatrix} \lambda_i & 0 \\ 0 & \frac{1}{\lambda_i} \end{bmatrix}$ in "$ad-bc=1$"-form

$\Rightarrow \left(\lambda_1 + \frac{1}{\lambda_1}\right)^2 = \left(\lambda_2 + \frac{1}{\lambda_2}\right)^2 \Leftrightarrow \lambda_1 + \frac{1}{\lambda_1} = \lambda_2 + \frac{1}{\lambda_2}$

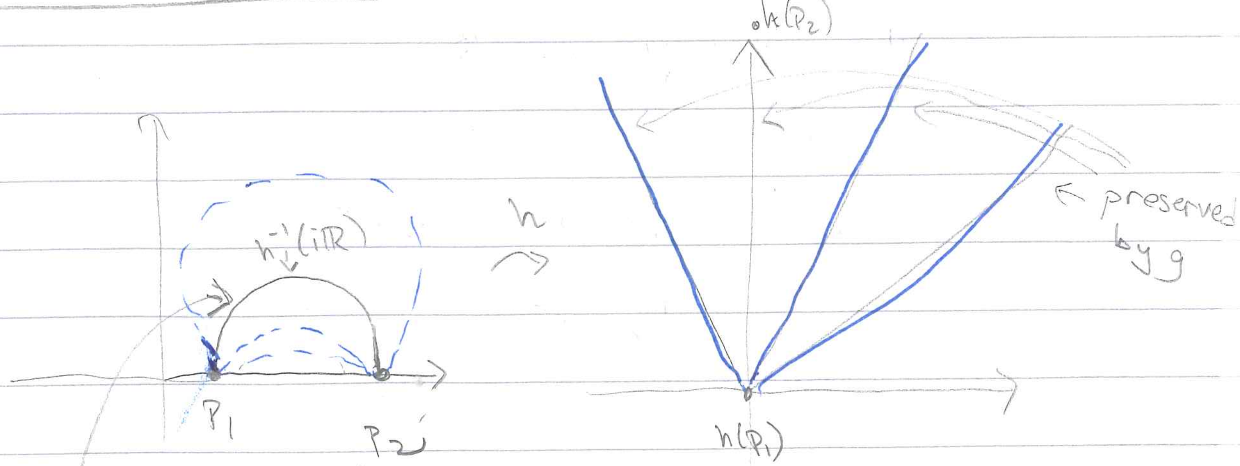
$\Leftrightarrow \lambda_1 - \lambda_2 = \dfrac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2} \Leftrightarrow (\lambda_1 - \lambda_2)\left(1 - \frac{1}{\lambda_1 \lambda_2}\right) = 0$

$\Rightarrow \underbrace{\lambda_1 = \lambda_2}_{\Rightarrow \eta_1 = \eta_2}$ or $\frac{1}{\lambda_1 \lambda_2} = 1 \Rightarrow \lambda_1 = \frac{1}{\lambda_2}$

$\Rightarrow \eta_1 = \frac{1}{\eta_2}$. This proves case 2.

[$g(z)$ inv. under conj. by $-\bar{z}$ $\Rightarrow$ 1-1 corresp between conjugacy classes of hyperbolic elements & real nbrs >1 in both $Möb^+(\mathbb{H})$ & $Möb(\mathbb{H})$]

Geometrically :



preserved by f since $i\mathbb{R}$ preserved by g

"the axis of f"

discuss **Def** $f \in \text{Möb}^+(\mathbb{H})$ of hyperbolic type w/ axis ℓ is called a "translation along ℓ"

Case 3 $\tau(f) < 4 \Rightarrow f$ has 2 fixpts p & $q \in \mathbb{C} \setminus \bar{\mathbb{R}}$ w/ $\bar{p} = q$. Assume $p \in \mathbb{H}$ ($q \notin \mathbb{H}$).

Q? Fin this

Let $h(z) = \dfrac{z - \text{Re}\,p}{\text{Im}\,p}$ $\Rightarrow a = 1, b = -\text{Re}\,p, c = 0, d = \text{Im}\,p$ $\Rightarrow ad - bc = \text{Im}\,p > 0$

Q $h \in \text{Möb}^+(\mathbb{H})$

$\Rightarrow h(p) = i \Rightarrow g = h \circ f \circ h^{-1} \in \text{Möb}^+(\mathbb{H})$ w/ $g(i) = i$.

Write $g(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha\delta - \beta\gamma = 1$

$g(i) = i \Rightarrow -\gamma = \beta$, $\delta = \alpha \Rightarrow \alpha^2 + \beta^2 = 1$

$\Rightarrow$ can write $\alpha = \cos(\theta)$, $\beta = \sin(\theta)$, for some $\theta \in \mathbb{R}$ and

$$g(z) = g_\theta(z) = \frac{\cos(\theta) z + \sin(\theta)}{-\sin(\theta) z + \cos(\theta)}$$

Remarks * If g_θ & $g_{\theta'}$ conjugate in $\text{Möb}^+(\mathbb{H})$, then $g_\theta = g_{\theta'}$

Q * $g_\theta = g_{\theta + \pi}$

So assume $g_{\theta'} = h^{-1} g_\theta h$ for some $h \in \text{Möb}^+(\mathbb{H})$.

$g_{\theta'}$ has i as fixed point $\Rightarrow g_\theta$ has $h(i)$ as fixed point

$h(i) \in \mathbb{H} \Rightarrow h(i) = i$ since i only fixpt of

g_θ in $\mathbb{H}$ $\Rightarrow h = g_\phi$ for some ϕ.

$\underline{Q}$ $g_\phi g_\theta = g_{\phi+\theta}$ $(\Rightarrow g_\phi g_\theta = g_\theta g_\phi)$.

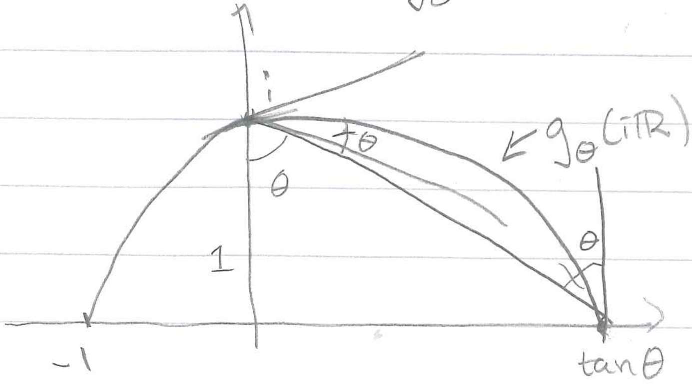
$\Rightarrow g_\phi^{-1} g_\theta g_\phi = g_\phi^{-1} g_\phi g_\theta = g_\theta \Rightarrow g_\theta = g_{\theta'}$

[Conj. classes in $\text{Möb}(\mathbb{H})$: If $h(z) = -\bar{z} \Rightarrow$
$h^{-1} g_\theta h = g_{-\theta} = g_{\pi-\theta} \Rightarrow$ get 1-1 corresp. w/ $\theta \in (0, \pi/2]$]

Geometrically:

Recall $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ describes a clock-wise rotation by the angle θ in $\mathbb{R}^2$.

Can also think of g_θ as rotation in $\mathbb{H}$



$g_\theta(i\mathbb{R})$: $g_\theta(0) = \frac{\sin\theta}{\cos\theta} = \tan\theta$ $g_\theta(\infty) = -1$ $\Rightarrow$ rotates imaginary axis by 2θ.

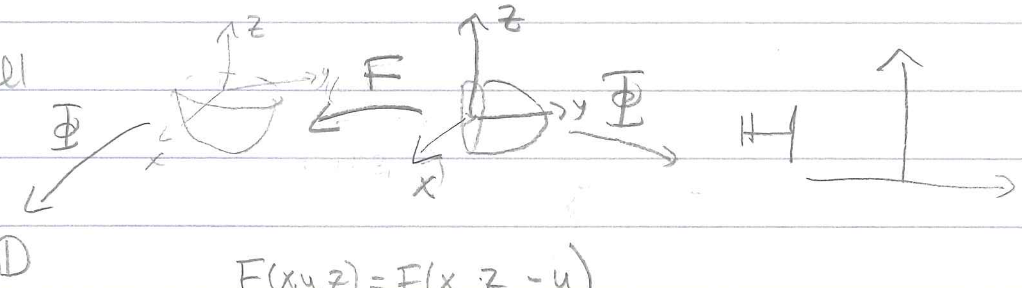
$g_\theta g_{\theta'} = g_{\theta+\theta'}$ $\Rightarrow$ g_θ rotates any $\mathbb{H}$ line through i by an angle 2θ

[rotate $i\mathbb{R}$ by $2(\theta+\theta')$ is same as first rotate g_θ $i\mathbb{R}$ to $g_\theta(i\mathbb{R})$ & then apply $g_{\theta'}$. □

Get an even better geometric picture if we go to $\mathbb{D}$:

§ 2.7 Poincaré's disk model $\mathbb{D}$

Recall



$F(x,y,z) = F(x, z, -y)$

$\Rightarrow$ $G = \Phi \circ F \circ \Phi^{-1} : \mathbb{H} \longrightarrow \mathbb{D}$ is a homeo

Q $G(u,v) = \left(\frac{2u}{u^2+(v+1)^2}, \frac{u^2+v^2-1}{u^2+(v+1)^2} \right)$ $\underset{\downarrow}{z = u+iv}$

$$= \frac{z+\bar{z}+i(z\bar{z}-1)}{|z+i|^2} = \frac{(iz+1)(\bar{z}-i)}{(z+i)(\bar{z}-i)} = \boxed{\frac{iz+1}{z+i}}$$

an FLT s.t. $G(0) = -i$, $G(1) = 1$, $G(-1) = -1$.

Exc from PSS: Use Cor 2.2.5 to derive this formula for G.

<u>Rmk</u> Many other possible FLT's identifying $\mathbb{H} \,\&\, \mathbb{D}$, but we use G since so simple.

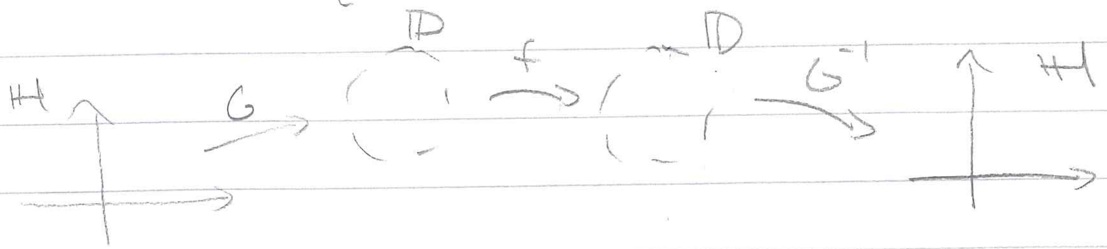
<u>Know</u> (from before) $\mathbb{D}$-lines $= \begin{cases} * \text{ diameters} \\ * \text{ circular arcs (which) } \cap \partial D \perp \end{cases}$

<u>Q</u> Can also be seen as a consequence of

$\Phi(\overline{\mathbb{R}}) = S^1 (= \partial D)$

Now consider Möbius transformations in this model:

Let $\text{Möb}(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{D} \mid G^{-1} f G \in \text{Möb}(\mathbb{H}) \}$



$\Rightarrow$ elements in $\text{Möb}(\mathbb{D})$ preserve angles & $\mathbb{D}$-lines.

<u>Q</u> Why?

Sim $\text{Möb}(\mathbb{D})^{\pm} = \{ f \in \text{Möb}(\mathbb{D}) \mid G^{-1} f G \in \text{Möb}^{\pm}(\mathbb{H}) \}$

Now we investigate the elements in $\text{Möb}^{+}(\mathbb{D})$ further, using matrices:

$G \longleftrightarrow \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$ $\quad G^{-1}(z) = \frac{iz-1}{-z+i}$ [use our formula from L:4]

$\Rightarrow G^{-1} \longleftrightarrow \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$

$\Rightarrow$ If $g(z) = \frac{az+b}{cz+d}$ w/ $a,b,c,d \in \mathbb{R}$, then

GgG^{-1} corresponds to

$$\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix} = \dots = \begin{bmatrix} -(a+d)+(c-b)i & -(b+c)-(a-d)i \\ -(b+c)+(a-d)i & -(a+d)-(c-b)i \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}$$ w/ $\alpha = -(a+d)+(c-b)i$

$\beta = -(b+c)-(a-d)i$

If now $g = g_\theta = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$ we get

$\alpha = -2\cos\theta - 2i\sin\theta = -2e^{i\theta}$ $\quad \Rightarrow GgG^{-1} = \frac{-2e^{i\theta} z}{-2e^{-i\theta}} = e^{i2\theta} z$

$\beta = 0$

which is nothing but rotation by 2θ.