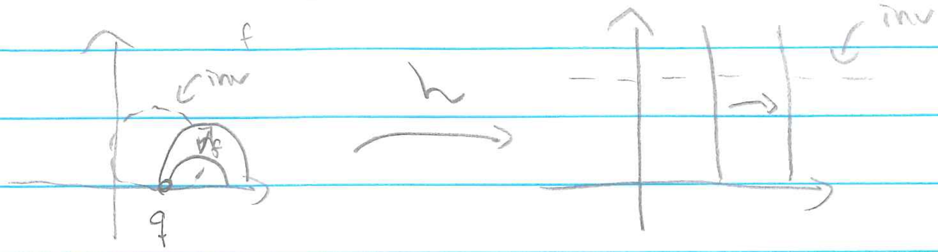
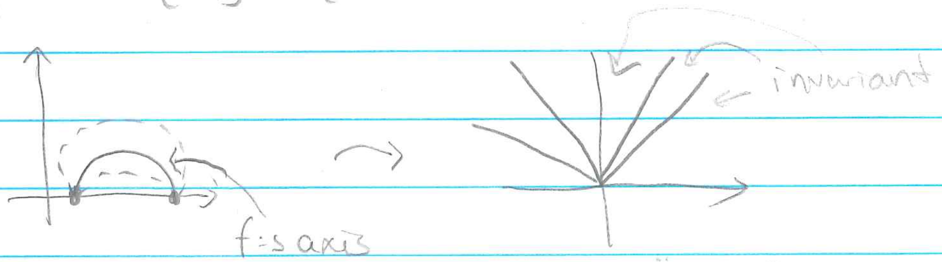
## Last time

$f(z) = \frac{az+b}{cz+d} \in Möb^+(\mathbb{H}) \rightsquigarrow \tau(f) = (a+d)^2$ invariant under conj. $hfh^{-1}$, $h \in Möb(\mathbb{H})$, and $f$ is conjugate to

* $z \mapsto z \pm 1$, if $\tau(f) = 4$ (one fixpt, real) "parabolic"




* $z \mapsto \eta z$, $\eta \in \mathbb{R}_{>1}$ if $\tau(f) > 4$ (2 real fixpts) "hyperbolic"

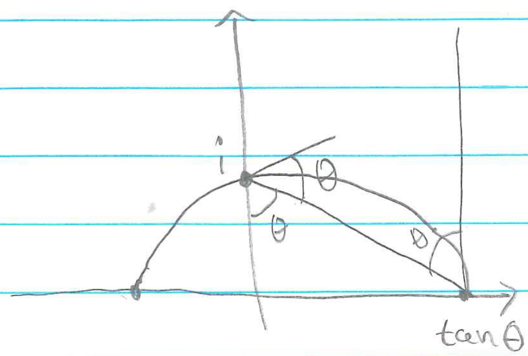



"translation" along this (this is inv)

* $z \mapsto g_\theta(z) = \frac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$, $\theta \in (0, \pi)$, if $\tau(f) < 4$ (2 cplx fixpts, only one of them in $\mathbb{H}$)

"elliptic"

Geometrically:




$0 \mapsto \tan\theta$

$\infty \mapsto -1$

$\Rightarrow i\mathbb{R}_{>0} \mapsto$ $\mathbb{H}$-line w/ endpts $-1$ & $\tan\theta$

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$g_\theta g_{\theta'} = g_{\theta+\theta'}$ $\Rightarrow$ $g_\theta$ rotates any $\mathbb{H}$ line through $i$ by an angle $2\theta$

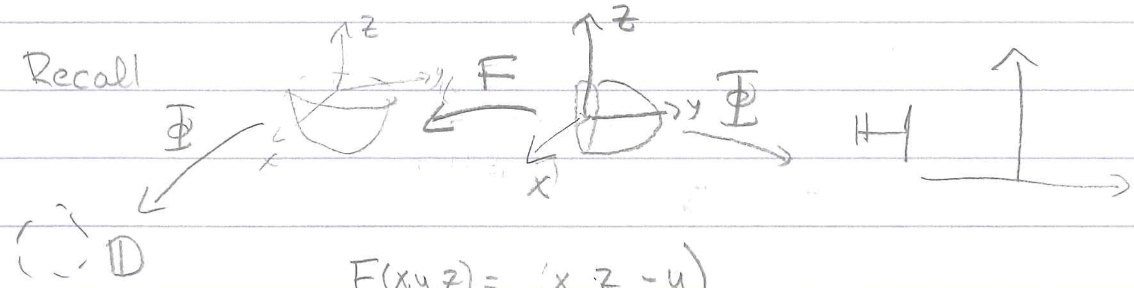
[ rotate $i\mathbb{R}$ by $2(\theta+\theta')$ is same as first rotate $i\mathbb{R}$ to $g_\theta(i\mathbb{R})$ & then apply $g_{\theta'}$. ]

$\square$

Get an even better geometric picture if we go to $\mathbb{D}$:

## § 2.7 Poincaré's disk model $\mathbb{D}$

Recall




$F(x,y,z) = (x, z, -y)$

$\Rightarrow$ $G = \Phi \circ F \circ \Phi^{-1} : \mathbb{H} \longrightarrow \mathbb{D}$ is a homeo

Q $G(u,v) = \left( \dfrac{2u}{u^2+(v+1)^2}, \dfrac{u^2+v^2-1}{u^2+(v+1)^2} \right)$ $\underset{z=u+iv}{\downarrow}$

$$= \frac{z+\bar{z}+i(z\bar{z}-1)}{|z+i|^2} = \frac{(iz+1)(\bar{z}-i)}{(z+i)(\bar{z}-i)} = \boxed{\frac{iz+1}{z+i}}$$

an FLT s.t. $G(0) = -i$, $G(1) = 1$, $G(-1) = -1$.

Exc from PSS: Use Cor 2.2.5 to derive this formula for $G$.

Rmk Many other possible FLT's identifying $\mathbb{H} \not\simeq \mathbb{D}$, but we use $G$ since so simple

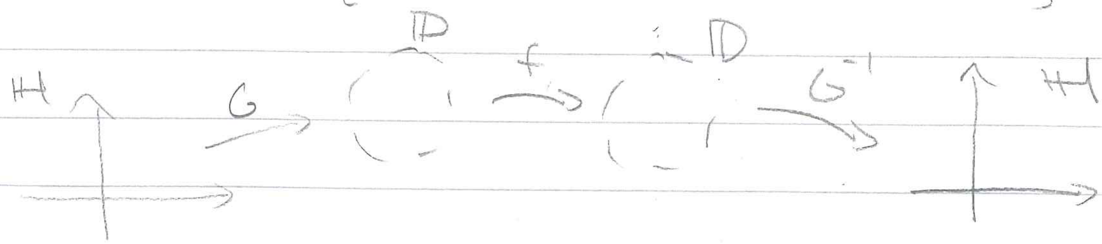
Know $\mathbb{D}$-lines = $\begin{cases} * \text{ diameters} \\ * \text{ circular arcs which } \cap \partial D \perp \end{cases}$
(from before)

Q Can also be seen as a consequence of

$G(\overline{\mathbb{R}}) = S^1 (= \partial D)$

Now consider Möbius transformations in this model:

Let $\text{Möb}(\mathbb{D}) = \{ f: \mathbb{D} \to \mathbb{D} \mid G^{-1} f G \in \text{Möb}(\mathbb{H}) \}$




$\Rightarrow$ elements in $\text{Möb}(\mathbb{D})$ preserve angles & $\mathbb{D}$-lines.

Q Why?

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L [illegible] : [illegible]

Sim $\text{Möb}(\mathbb{D})^{\pm} = \{ f \in \text{Möb}(\mathbb{D}) \mid G^{-1} f G \in \text{Möb}^{\pm}(\mathbb{H}) \}$

Now we investigate the elements in $\text{Möb}^{+}(\mathbb{D})$ further, using matrices:

$G \longleftrightarrow \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$ $\qquad G^{-1}(z) = \dfrac{iz-1}{-z+i}$ [use our formula from L 4]

$\Rightarrow G^{-1} \longleftrightarrow \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$

$\Rightarrow$ If $g(z) = \dfrac{az+b}{cz+d}$ w/ $a,b,c,d \in \mathbb{R}$, then

$GgG^{-1}$ corresponds to

$$\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix} = \ldots = \begin{bmatrix} -(a+d)+(c-b)i & -(b+c)-(a-d)i \\ -(b+c)+(a-d)i & -(a+d)-(c-b)i \end{bmatrix}$$

$$= \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \quad \text{w/ } \alpha = -(a+d)+(c-b)i$$

$$\beta = -(b+c)-(a-d)i$$

If now $g = g_\theta = \dfrac{\cos\theta z + \sin\theta}{-\sin\theta z + \cos\theta}$ we get

$\alpha = -2\cos\theta - 2i\sin\theta = -2e^{i\theta}$ $\qquad \Rightarrow GgG^{-1} = \dfrac{-2e^{i\theta} z}{-2e^{-i\theta}} = e^{i2\theta} z$

$\beta = 0$

which is nothing but rotation by $2\theta$.

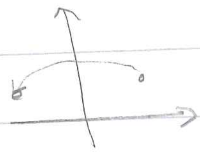
# Classification of elements in $Möb^-(\mathbb{H})$

$a,b,c,d \in \mathbb{R}$

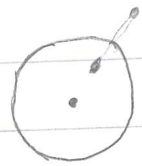
$f \in Möb^-(\mathbb{H}) \iff f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$, $ad-bc=-1$

Special cases: • $f(z) = -\bar{z}$

= horizontal reflection in the imaginary axis



• $f(z) = \frac{1}{\bar{z}}$ = "inversion" in the circle $|z|=1$:



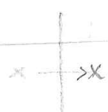
$z = re^{i\theta} \mapsto \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta}$

$\Rightarrow$ point outside circle $\leftrightarrow$ point inside circle s.t. angle w/ $\mathbb{R}$-axis preserved

$f|_{|z|=1} = id$

**Def** An inversion in the circle $C=\{|z-m|=r\}$ is a map $g:\mathbb{C}\to\mathbb{C}$ s.t. $|g(z)-m||z-m|=r^2$.

An inversion in a vertical line $\ell$ $=\{x=x_0\}$ is horizontal reflection in $\ell$.



$z \mapsto -\bar{z}+2x_0$

**Prp 2.3.3** Let $f \in \text{Möb}^-(\mathbb{H})$, $f(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$,

$a, b, c, d \in \mathbb{R}$, $ad - bc = -1$.

- If $a+d=0$, then $f$ is an inversion, conjugate to $-\bar{z}$.
- If $a+d \neq 0$, then $f = gh$, where $g$ is an inversion, $h \in \text{Möb}^+(\mathbb{H})$ of hyperbolic type w/ axis equal to the line of inversion of $g$. Moreover, $g$ and $h$ commute and this decomposition is unique (given $gh = hg$)

Pf ~~Also~~ in this case we consider the fixpts of $f$:

$$z = \frac{\bar{z}a+b}{c\bar{z}+d} \iff c\underbrace{|z|^2}_{\substack{z\bar{z} \\ \shortparallel \\ x^2+y^2}} + dz - a\bar{z} - b = 0$$

$\iff$ (*) $c(x^2+y^2) - (a-d)x - b = 0$ (Real part)

~~(**)~~ $(a+d)y = 0$ (Imaginary part)

$(f(\infty) = \infty \iff c = 0)$

Case 1 $a+d=0 \Rightarrow$ only the Realpart-eqn

$\underline{c=0}$: This gives a vertical $\mathbb{H}$-line $\{x = x_0\}$ $\quad x_0 = \frac{b}{a-d}$

Let $h(z) = z - x_0 \Rightarrow g = h \circ f \circ h^{-1}$ has fixpts

$=$ imaginary axis

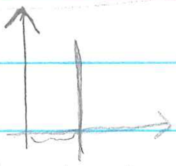
$\underline{Q} \Rightarrow g(z) = -\bar{z}$

$$i = \frac{-ai+b}{-ci+d} \qquad c = b, \quad d = -a \qquad \Rightarrow b = c = 0$$

$$2i = \frac{-2ai+b}{-2ci+d} \qquad 4c = b, \quad 2d = -2a \qquad ad = -1$$

$$\Rightarrow a = -d = \pm 1$$

$\Rightarrow f = h^{-1} \circ g \circ h = (-(\overline{z - x_0})) + x_0 = -\bar{z} + 2x_0$



$\{x = x_0\}$

which is nothing but inversion in $\{x = x_0\}$.

$\underline{c \neq 0}$: Then the fixpoints form a semi-circle w/

center $m = (a/c, 0)$ & radius $r = 1/|c|$ (completing the square).

$$f(z) = \frac{a}{c} + \frac{1/c}{c\bar{z}+d} = \left[\frac{c\frac{a}{c}\bar{z} + \frac{ad+1}{c}}{c\bar{z}+d}\right] = \underbrace{\frac{a}{c}}_{m} + \frac{\overbrace{1/c^2}^{r^2}}{\bar{z} - a/c}$$

$-b$ since $ad - bc = -1$

$$= m + \frac{r^2}{\bar{z} - m} = m + r^2 \frac{z-m}{|z-m|^2}$$

$$\Rightarrow |f(z) - m||z - m| = r^2 \frac{|z-m|}{|z-m|^2}|z-m| = r^2$$

$\Rightarrow f(z)$ is an inversion in its fixpointset

Q $f$ conjugate to $z \mapsto -\bar{z}$

[ $h$ map $C$ to $i\mathbb{R}_{>0}$ $\Rightarrow_{g = h \circ f \circ h^{-1}}$ has $i\mathbb{R}$ as fixset
$\Rightarrow g = -\bar{z}$ ]

$\triangle$ Case $a+d=0$

$a+d \neq 0$ $(**) \Rightarrow y=0 \Rightarrow$ no fixpts in $\mathbb{H}$

Claim $\exists$ a $\mathbb{H}$-line $l$ s.t. $f(l)=l$ as sets, and the fixpts of $f$ = the endpoints of $l$

Pf If $c=0$ then $f(\infty)=\infty$, $f\left(\underbrace{\frac{b}{d-a}}_{\neq 0 \text{ since } da=-1}\right) = \frac{b}{d-a}$

$\Rightarrow f\left(\left\{x = \frac{b}{d-a}\right\}\right) = \left\{x = \frac{b}{d-a}\right\}$ since this is the $\mathbb{H}$-line w/ $\infty$ & $\frac{b}{d-a}$ as endpoints.

$c \neq 0$, then $f$ has 2 real fixpts

$x_{\pm} = \frac{a-d \pm \sqrt{(a+d)^2+4}}{2c}$, and hence $f$ must map the $\mathbb{H}$-line $l$ w/ these 2 pts as endpts to itself.

$\triangle$ claim

Let $h \in \text{Möb}^+(\mathbb{H})$ s.t. $h(l)$ = imaginary axis

$\Rightarrow g = h \circ f \circ h^{-1}$ maps imaginary axis $\mapsto$ imaginary axis

Q

$\Rightarrow g(z) = -\lambda^2 \bar{z}$ $\quad$ ( $-\overline{g(z)}$ maps $i\mathbb{R}$ to $i\mathbb{R}$ of the form $\frac{az+b}{cz+d}$ & $\lambda^2 z$ satisfies this $\Rightarrow g(z) = -\lambda^2 \bar{z}$ )

and this is a composition of $\tilde{g} = -\bar{z}$ & $\tilde{h}(z) = \lambda^2 z$

$\Rightarrow$ $\tilde{g}$ inversion in $i\mathbb{R}$

& $\tilde{h}$ is a hyperbolic transformation w/ axis $i\mathbb{R}$.

Also $g = \underbrace{\tilde{g} \circ \tilde{h}}_{= -(\overline{\lambda^2 z})} = \underbrace{\tilde{h} \tilde{g}}_{= \lambda^2(-\bar{z})}$ so they commute.

Conjugate back $\Rightarrow$ $f = h^{-1} g h = h^{-1} \tilde{g} \tilde{h} h = \underbrace{(h^{-1} \tilde{g} h)}_{\text{inversion in line } h^{-1}(i\mathbb{R})} \underbrace{(h^{-1} \tilde{h} h)}_{\text{hyperbolic w/ axis } h^{-1}(i\mathbb{R})}$

Uniqueness See book. $\square$

⚠ For uniqueness we need $g$ & $h$ to commute, otherwise many ways of writing $f \in \text{Möb}^-(\mathbb{H})$ as $f = gh$ w/ $g$ inversion, $h \in \text{Möb}^+(\mathbb{H})$.

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<u>Cor</u> * $f \in Möb^-(\mathbb{H})$ inversion $\Leftrightarrow$ $a+d=0$

* $f \in Möb^-(\mathbb{H})$ inversion $\Leftrightarrow$ it has a fixpt in $\mathbb{H}$

* $f \in Möb(\mathbb{H})$ inversion in $l$

$\Leftrightarrow$ has $l$ as fixpointset.