Last time classified elements in $\text{Möb}^-(\mathbb{H})$:

inversion in its fixpointset if has fixpts in $\mathbb{H}$

inversion $\circ$ $\underbrace{\text{hyperbolic transformation}}_{z \mapsto \eta z,\ \eta > 1}$ if no fixpts in $\mathbb{H}$.

Also considered $f \in \text{Möb}^+(\mathbb{D}) = \{ f : \mathbb{D} \to \mathbb{D} \mid G^{-1} f G \in \text{Möb}^+(\mathbb{H}) \}$

$G : \mathbb{H} \to \mathbb{D}$, $\quad G(z) = \dfrac{iz+1}{z+i}$

Using matrices we saw $f \in \text{Möb}^+(\mathbb{D}) \Rightarrow$

$$f = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \qquad \left.\begin{aligned} \alpha &= -(a+d) + (c-b)i \\ \beta &= (b+c) - (a-d)i \end{aligned}\right\} (*)$$

since
$$\underbrace{\underbrace{\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}}_{G}}_{\det = -2} \underbrace{\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{G^{-1}fG}}_{\det > 0} \underbrace{\underbrace{\begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}}_{G^{-1}}}_{\det = -2} = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}$$

$$\Rightarrow \alpha\bar{\alpha} - \beta\bar{\beta} = 4(ad - bc)$$

<u>Note</u> given $\alpha, \beta \in \mathbb{C}$ $\exists\, a, b, c, d \in \mathbb{R}$ satisfying $(*)$

$$\Rightarrow f(z) = \frac{\alpha z + \beta}{\bar\beta z + \bar\alpha} \quad \text{gives element in } \text{Möb}^+(\mathbb{D})$$

$$\Longleftrightarrow \alpha\bar\alpha - \beta\bar\beta > 0$$

Normalizing by a <u>real</u> nbr $\Rightarrow$ might assume

$\alpha\bar\alpha - \beta\bar\beta = 1$

$$\Rightarrow \text{Möb}^+(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{D} \;\middle|\; f = \frac{\alpha z + \beta}{\bar\beta z + \bar\alpha},\ \alpha, \beta \in \mathbb{C},\ \alpha\bar\alpha - \beta\bar\beta = 1 \right\}$$

$$\underline{\text{Möb}^-(\mathbb{D})} = \{ f : \mathbb{D} \to \mathbb{D} \mid G^{-1} f G \in \text{Möb}^-(\mathbb{H}) \}$$

<u>Q</u> $*$ $\text{Möb}^-(\mathbb{H}) = \text{Möb}^+(\mathbb{H}) \circ \frac{1}{\bar z}$ $\left[\frac{\frac{a}{\bar z} + b}{\frac{c}{\bar z} + d} = \frac{a + b\bar z}{c + d\bar z} \right]$

$*$ $G \circ \frac{1}{\bar z} \circ G^{-1} = \bar z$ $\left[\frac{iz+1}{-z+i} \mapsto \frac{-\bar z - i}{-i\bar z - 1} \mapsto \frac{i\left(\frac{-\bar z - i}{-i\bar z - 1}\right) + 1}{\frac{-\bar z - i}{-i\bar z - 1} + i} \right]$

$$\Rightarrow \text{Möb}^-(\mathbb{D}) = \text{Möb}^+(\mathbb{D}) \circ \bar z$$

$$= \left\{ f : \mathbb{D} \to \mathbb{D} \;\middle|\; f = \frac{\alpha \bar z + \beta}{\bar\beta \bar z + \bar\alpha},\ \alpha\bar\alpha - \beta\bar\beta = 1 \right\}$$

<u>Ex</u> • Complex conjugation in $\mathbb{D} \Leftrightarrow$ inversion in $|z| = 1$ in $\mathbb{H}$

<u>Q</u> • Reflection in $i\mathbb{R}$ in $\mathbb{D} \Leftrightarrow$ Reflection in $i\mathbb{R}$ in $\mathbb{H}$

§2.4 Hilbert's axioms & congruence in $\mathbb{H}$

Aim Show $\mathbb{H}$ w/ $\mathbb{H}$-lines gives a model for the hyperbolic plane.

Incidence - maybe on oblig...

Betweenness Define this as

$x*y*z \iff x,y,z$ on same $\mathbb{H}$-line ℓ and

$(\operatorname{Im}x - \operatorname{Im}y)(\operatorname{Im}y - \operatorname{Im}z) > 0$ if ℓ is vertical

$(\operatorname{Re}x - \operatorname{Re}y)(\operatorname{Re}y - \operatorname{Re}z) > 0$ if ℓ semi-circel.

Q Draw pictures of this.

Can check that B1-B4 then hold. Can also show that the continuity axiom holds.

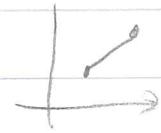
For congruence we will use $\text{Möb}(\mathbb{H})$.

First some notation:

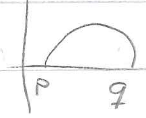
If $z_1, z_2 \in \mathbb{H}$, then let $\overleftrightarrow{z_1 z_2}$ be the $\mathbb{H}$-line containing them

$\overrightarrow{z_1 z_2}$ - ray from z_1 containing z_2

$[z_1, z_2]$ - Segment between z_1 & z_2.



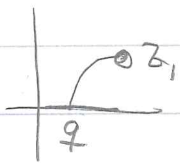
- $(p,q) = l$ the $\mathbb{H}$-line w/ endpts $p, q \in \overline{\mathbb{R}}$.



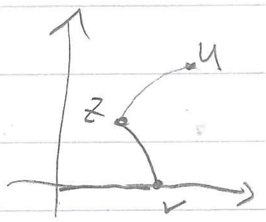
$[z_1, q) = \overrightarrow{z_1 q}$ the ray from z_1 w/ endpoint q

vertex of the ray

endpoint of the ray



$\angle uzv = \{\overrightarrow{zu}, \overrightarrow{zv}\}$ unordered, $z \in \mathbb{H}$,
u, v in $\mathbb{H}$ or $\overline{\mathbb{R}}$



Define the congruence relation in $\mathbb{H}$ as follows:

Congruence of segments

$[z_1, z_2] \cong [w_1, w_2] \iff g([z_1, z_2]) = [w_1, w_2]$

for some $g \in \text{Möb}(\mathbb{H})$

Congruence of angles

$\angle uzv \cong \angle u'z'v' \iff g(\overrightarrow{zu}) = \overrightarrow{z'u'}, g(\overrightarrow{zv}) = \overrightarrow{z'v'}$

for some $g \in \text{Möb}(\mathbb{H})$ (Notation $g(\angle uzv) = \angle u'z'v'$)

Recall the axioms

C1 Given a segment $[z_1, z_2]$ & a ray σ w/ vertex w_1, $\exists!\ w_2 \in \sigma$ s.t. $[w_1, w_2] \cong [z_1, z_2]$

C2 $\cong$ equivalence relation on {segments}

C3 If $z_1 * z_2 * z_3$ & $w_1 * w_2 * w_3$ and both $[z_1, z_2] \cong [w_1, w_2]$ & $[z_2, z_3] \cong [w_2, w_3]$ then also $[z_1, z_3] \cong [w_1, w_3]$

C4 Given a ray $[w, q)$ and an angle $\angle uzv$, there are unique angles $\angle p_1 w q$ & $\angle p_2 w q$ on opposite sides of $[w, q)$ s.t. $\angle p_1 w q \cong \angle p_2 w q \cong \angle uzv$.

C5 $\cong$ equivalence rel on {angles}

C6 SAS: Given triangles $z_1 z_2 z_3$ & $w_1 w_2 w_3$. If $[z_1, z_2] \cong [w_1, w_2]$, $[z_1, z_3] \cong [w_1, w_3]$ and $\angle z_2 z_1 z_3 \cong \angle w_2 w_1 w_3$, then the two triangles are congruent.

C2, C5 follows since congruence is def. by a gp action.

For the other ones, need:

Lma 2.4.2 Suppose $z_j \in l_j \subset \mathbb{H}$ -line w/ endpts p_j & q_j, $j=1,2$. Then $\exists!\, f \in \text{Möb}^+(\mathbb{H})$ s.t. $f(p_1)=p_2$, $f(q_1)=q_2$, $f(z_1)=z_2$, hence also $f(l_1)=l_2$

Pf Exc 2.2.7 b) from PSS 1. Alt. see pf in book □

$\Rightarrow f([z_1, q_1)) = [z_2, q_2)$

Cor 2.4.3 $\text{Möb}^+(\mathbb{H})$ acts transitively on the set of all rays. In fact, given 2 rays σ_1 & σ_2 w/ vertices z_1 & z_2, $\exists!\, f \in \text{Möb}^+(\mathbb{H})$ s.t. $f(z_1)=z_2$ & $f(\sigma_1)=\sigma_2$)

(Complete rays to $\mathbb{H}$-lines & apply Lma 2.4.2)

Lma 2.4.4. (i) An element in $M\ddot{o}b^+(\mathbb{H})$ is completely determined by its values at 2 pts in $\mathbb{H}$.

(ii) Suppose $[z_1, z_2] \cong [w_1, w_2]$. Then $\exists!\ f \in M\ddot{o}b^+(\mathbb{H})$ s.t. $f(z_1) = w_1$ & $f(z_2) = w_2$.

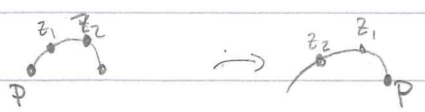
Lma 2.4.5 Given 2 rays σ_1 & σ_2 w/ a common vertex z_0. Then $\exists!$ inversion g s.t. $g(z_0) = z_0$, $g(\sigma_1) = \sigma_2$ & $g(\sigma_2) = \sigma_1$.

Pf of Lma 2.4.4

i) Cor 2.2.5 : values of f at 3 pts determine f uniquely.

But 2 (z_1, z_2) pts $\in \mathbb{H}$ uniquely determines a $\mathbb{H}$-line ℓ 6.

w/ endpts p & q. (Assume p s.t. $p * z_1 * z_2$) From the betweenness relations we see which of the endpts of $f(\ell)$ that corresponds to p.



$\Rightarrow$ know the values of f at 3 pts

$\Rightarrow$ f uniquely determined.

ii) Assume $g([z_1,z_2]) = [w_1,w_2]$ for some $g \in \text{Möb}(\mathbb{H})$

If $g \in \text{Möb}^-(\mathbb{H})$, let k be the inversion in $\overleftrightarrow{w_1 w_2}$

$\Rightarrow \tilde{g} = k \circ g \in \text{Möb}^+(\mathbb{H})$ w/ $\tilde{g}([z_1,z_2]) = [w_1,w_2]$

If $\tilde{g}(z_1) = w_1$, $\tilde{g}(z_2) = w_2$ we are done.

Otherwise $\tilde{g}(z_1) = w_2$, $\tilde{g}(z_2) = w_1$. Then let

$h \in \text{Möb}^+(\mathbb{H})$ s.t. $h(\overleftrightarrow{w_1 w_2}) = i\mathbb{R}_{>0}$

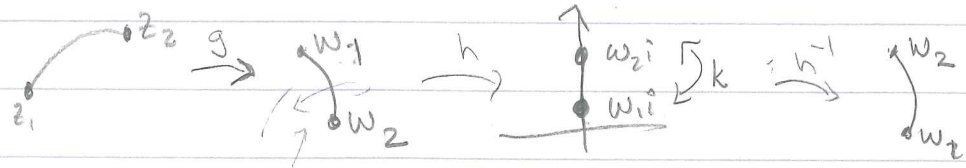
$\Rightarrow \exists\, \omega_1, \omega_2 \in \mathbb{R}_{>0}$ s.t. $h(w_1) = \omega_1 i$, $h(w_2) = \omega_2 i$

Let $k(z) = \dfrac{-\omega_1 \omega_2}{z}$ $\Rightarrow k(\omega_1 i) = \omega_2 i$, $k(\omega_2 i) = \omega_1 i$

$\Rightarrow h^{-1} k h(w_1) = w_2$, $h^{-1} k h(w_2) = w_1$

$\Rightarrow f = h^{-1} k h \tilde{g} \in \text{Möb}^+(\mathbb{H})$ maps z_1 to w_1, z_2 to w_2

or g if $g \in \text{Möb}^+(\mathbb{H})$



add inversion if $g \notin \text{Möb}^+(\mathbb{H})$

$\boxtimes$

Q: Uniqueness?

$\square$

Pf of Lma 2.4.5

Lma 2.4.2 $\Rightarrow \exists\ h \in Möb^+(\mathbb{H})$ s.t. $h(\sigma_1) = \sigma_2$

$\Rightarrow h(z_0) = z_0$ (& this is the only fixpt if $h \neq id \Rightarrow h$ elliptic)

Let r be inversion in the $\mathbb{H}$-line containing σ_1

Let $g = hr \in Möb^-(\mathbb{H}) \Rightarrow g(z_0) = z_0 \in \mathbb{H}$

$\Rightarrow g$ inversion (Cor from last time).

Also $g(\sigma_1) = hr(\sigma_1) = h(\sigma_1) = \sigma_2$

$g(\sigma_2) = g(g(\sigma_1)) = \sigma_1$ since g inversion

Uniqueness Suppose g' another inversion w/ same properties

$\Rightarrow g^{-1}g' \in Möb^+(\mathbb{H})$ s.t. $g^{-1}g'(\sigma_i) = \sigma_i \quad i = 1, 2$

$\Rightarrow g^{-1}g'$ has 3 fixpts given by the endpts of

σ_1 & σ_2 (can only have z_0 in common, otherwise coincides)

$\Rightarrow g^{-1}g' = id.$

Now we can show that the congruence axioms are satisfied.

C1 $[z_1, z_2] \rightsquigarrow \overrightarrow{z_1 z_2} \xRightarrow{\text{Cor } 2.4.3} \exists f \in \text{Möb}^+(\mathbb{H})$

s.t. $f(z_1) = w_1$, $f(\overrightarrow{z_1 z_2}) = \sigma$.

Choose $w_2 = f(z_2) \Rightarrow [w_1, w_2] \cong [z_1, z_2]$

Uniqueness of w_2: Assume $w_2' \in \sigma$, $w_2' \neq w_2$

s.t. $[w_1, w_2'] \cong [z_1, z_2]$.

Lma 2.4.4 $\Rightarrow \exists\, h \in \text{Möb}^+(\mathbb{H})$ s.t. $h(z_1) = w_1$

$h(z_2) = w_2' \Rightarrow h(\overrightarrow{z_1 z_2}) = \overrightarrow{w_1 w_2'} = \sigma$

$\xRightarrow[\text{in Cor } 2.4.3]{\text{Uniqueness}} h = f \Rightarrow w_2 = f(z_2) = h(z_2) = w_2'$ $\square$

C3 $[z_1, z_2] \cong [w_1, w_2] \xRightarrow{\text{Lma } 2.4.4 \text{ (ii)}} \exists\, g \in \text{Möb}^+(\mathbb{H})$

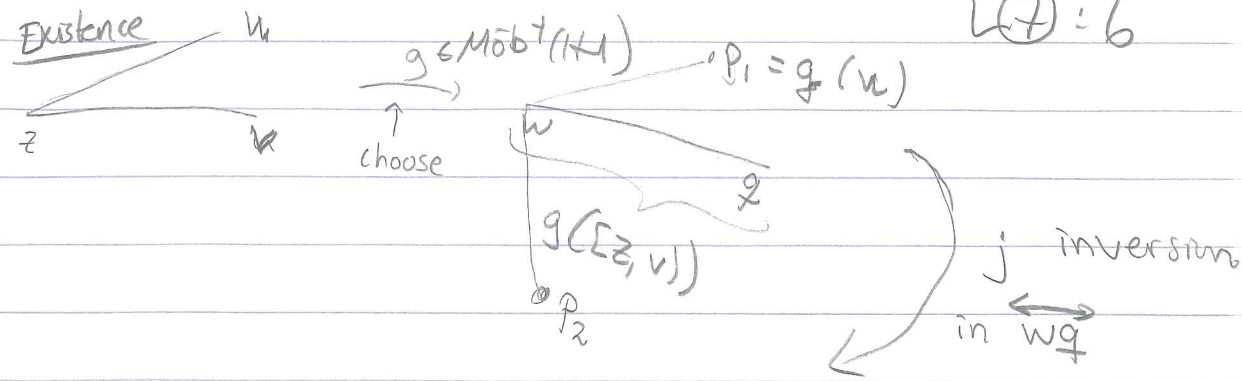
s.t. $g(z_1) = w_1$, $g(z_2) = w_2$

$\Rightarrow w_3' = \boxed{\overset{Q}{g(z_3)} \in \overrightarrow{w_2 w_3}}$ [g continuous] & g defines

congruences $[z_2, z_3] \cong [w_2, w_3']$ and $[z_1, z_3] \cong [w_1, w_3']$

Uniqueness in C1 $\Rightarrow w_3' = w_3$ $\square$

C4 Existence



$\angle p_2 w q = j\,g(\angle uzv) = j(\angle p_1 w q)$

Uniqueness Suppose $h(\angle uzv) = \angle pwq$

$h'(\angle uzv) = \angle p'wq$

s.t. p & p' on the same side of $[w,q]$, $h, h' \in Möb(\mathbb{H})$

$\Rightarrow h'h^{-1}(\angle pwq) = \angle p'wq$

$\Rightarrow h'h^{-1}(\overrightarrow{wq}) = \overrightarrow{wq}$ or $\overrightarrow{wp}$ $\Big]$ g inversion from Lma 7.4.5

$\Rightarrow$ may assume $h'h^{-1}(w) = w$

$h'h^{-1}(q) = q$

$\Rightarrow h'h^{-1}$ either id or $\underbrace{\text{reflection in } \overleftrightarrow{wq}}$

$\Downarrow$ p, p' on same side of $[w,q]$

$\Rightarrow h'h^{-1} = \mathrm{id} \Rightarrow \angle pwq = \angle p'wq$.

C6 By assumption $\angle w_2 w_1 w_3 = g(\angle z_2 z_1 z_3)$

for some $g \in$ Möb$(\mathbb{H})$.

Using $\tilde{g}$ from Lma 2.4.5 (if nec.)

$\rightsquigarrow$ may assume $g(\overrightarrow{z_1 z_2}) = \overrightarrow{w_1 w_2}$

$$g(\overrightarrow{z_1 z_3}) = \overrightarrow{w_1 w_3}$$

Uniqueness in C1 $\Rightarrow$ $g(z_2) = w_2$, $g(z_3) = w_3$

(since $[z_1, z_2] \cong [w_1, w_2]$, $[z_1, z_3] \cong [w_1, w_3]$

by assumption)

$\Rightarrow g([z_2, z_3]) = [w_2, w_3]$

$g(\angle z_1 z_2 z_3) = \angle w_1 w_2 w_3$

$g(\angle z_2 z_3 z_1) = \angle w_2 w_3 w_1$ $\square$

Rmk Except for C6 we could have def. congruence using Möb$^+(\mathbb{H})$