F(8) : 0

## Last time

Defined congruence in $\mathbb{H}$ and proved that Hilbert's axioms C1–C6 hold.

**Congruence of segments**

$[z_1, z_2] \cong [w_1, w_2] \iff \exists\, g \in \text{Möb}(\mathbb{H})$ s.t.

$g(z_1) = w_1, \; g(z_2) = w_2$

**Congruence of angles**

$\angle uzv \cong \angle u'z'v' \iff \exists\, g \in \text{Möb}(\mathbb{H})$ s.t.

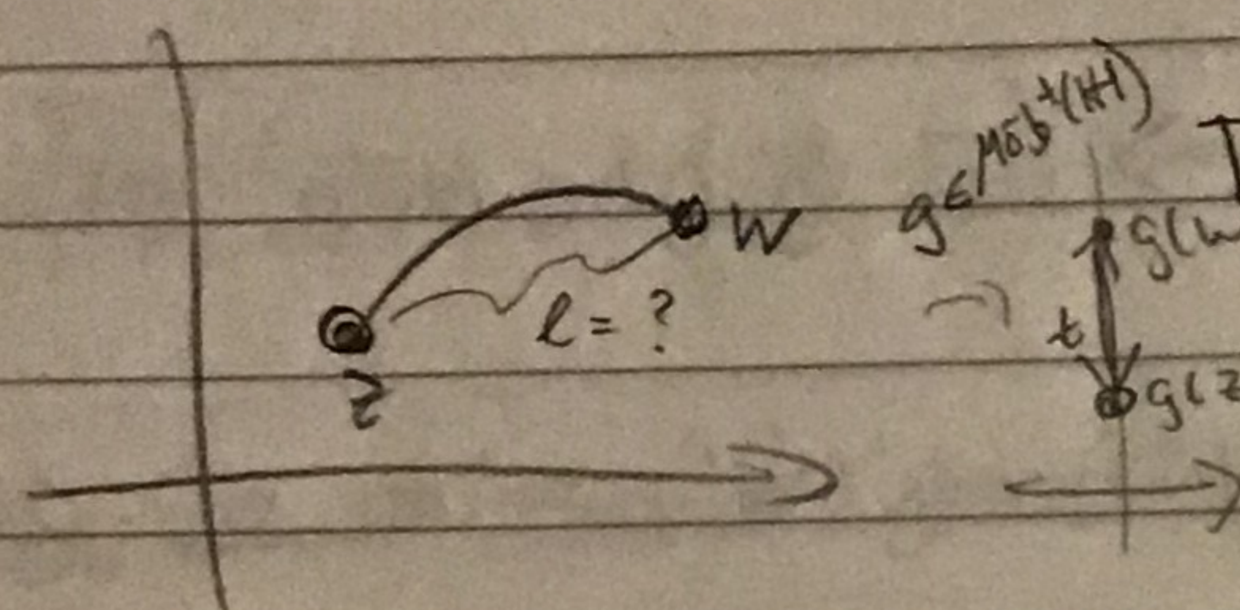
$g(\overrightarrow{zu}) = \overrightarrow{z'u'}, \; g(\overrightarrow{zv}) = \overrightarrow{z'v'}$.

**Rmk** Can define this also in $\mathbb{D}$, replacing $\text{Möb}(\mathbb{H})$ w/ $\text{Möb}(\mathbb{D})$, and the axioms will still hold.

F⑧:1

## §2.5 Distance in $\mathbb{H}$

To define the distance between $z, w \in \mathbb{H}$ we want to "measure the length" of the segment between them




To do this, map $\overleftrightarrow{zw}$ to $i\mathbb{R}_{>0}$ w/ $g \in \text{Möb}^+(\mathbb{H})$ s.t. $g(z) = i$

$\Rightarrow g(w) = ti$ for some $t \in \mathbb{R}_{>0}$

Want $d(z, w) \sim t$

Define $d(z,w) = \left| \ln |g(w)| \right|$

Q Well-defined (doesnot depend on g)

Q if $p, q$ endpts of $\overleftrightarrow{wz}$ then $d(z,w) = \left| \ln \left| \frac{w-p}{w-q} \frac{z-q}{z-p} \right| \right|$

Need to check that this is a metric, so should satisfy:

(d1) $d(z,w) \geq 0$ & $d(z,w) = 0 \iff z = w$.

But this is clear

(d2) $d(z,w) = d(w,z) \quad \forall z, w \in \mathbb{H}$ :

$g(z) = i \quad g \in \text{Möb}^+(\mathbb{H})$

Assume $g(w) = ti$ and define $h(u) = \frac{-t}{g(u)} \Rightarrow$

$h \in \text{Möb}^+(\mathbb{H})$, $h(w) = i$, $h(z) = ti \Rightarrow$

$d(w, z) = |\ln(t)| = d(z, w)$.

(d3) $d(z,w) \leq d(z,u) + d(u,w) \quad \forall\, z,u,w \in \mathbb{H}$:
(triangle ineq)

Before proving this, prove "measure distance along lines"-property

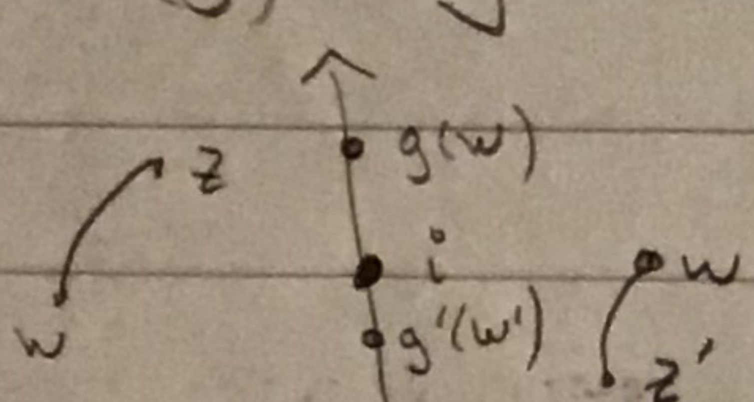
(d5) $\underbrace{d(z,w)}_{=|\ln(t)|} = d(z',w') \iff [z,w] \cong [z',w']$ :

$\Rightarrow \exists\, g, g' \in \text{Möb}^+(\mathbb{H})$ s.t.

$g(z) = g'(z') = i$, $\underbrace{g(w) = g'(w') = it}_{i)}$ or $\underbrace{g(w) = \frac{i}{t},\ g'(w') = it}_{ii)}$

[since $|\ln(t)| = |\ln(1/t)|$]

i): $h = (g')^{-1} g$ satisfies $h(z) = z'$, $h(w) = w'$
$\Rightarrow [z,w] = [z',w']$




ii): $h = (g')^{-1} \circ \frac{-1}{g}$ satisfies $h(z) = z'$, $h(w) = w'$
$\Rightarrow$ ok

Q Check this

$-\frac{1}{g(w)} = \frac{-1}{\frac{i}{t}} = -\frac{t}{i} = ti$

$\Delta$ "$\Rightarrow$"

"$\Leftarrow$" Then $h(z) = z'$, $h(w) = w'$ for some $h \in \text{Möb}(\mathbb{H})$

W.l.o.g $h \in \text{Möb}^+(\mathbb{H})$ after composing w/ inversion

in $\overleftrightarrow{zw}$

F⑧:2

Assume $g \in Möb^+(\mathbb{H})$ s.t. $g(\overleftrightarrow{zw}) = i\mathbb{R}_{>0}$, $g(z) = i$,

let $t = \frac{g(w)}{i}$ $\Rightarrow g' = g h^{-1} \in Möb^+(\mathbb{H})$ &

$g'(\overleftrightarrow{z'w'}) = i\mathbb{R}_{>0}$, $g'(z') = i$, $g'(w') = g(w) = ti$

$\Rightarrow d(z,w) = d(z', w')$. □

Now we prove a modified version of d3.

(d3+d4) $d(z,w) \le d(z,u) + d(u,w)$ $\forall z,u,w \in \mathbb{H}$.

If $z,u,w$ distinct then we have equality $\Leftrightarrow$

$u \in [z,w]$.

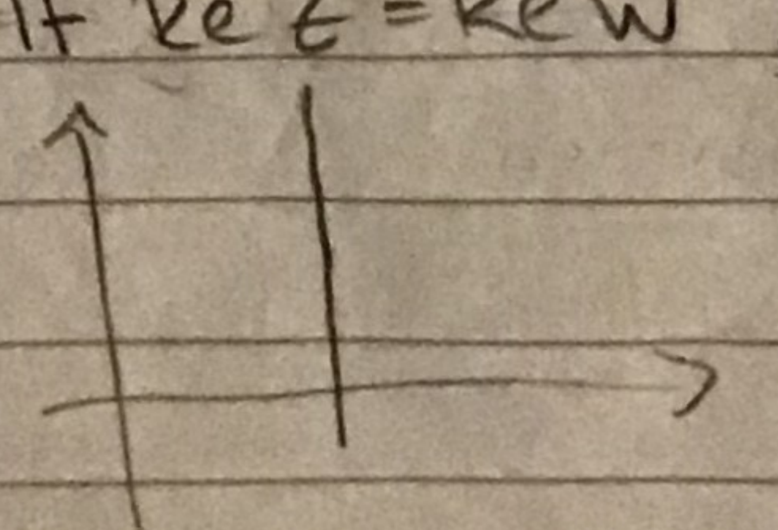
To do this, use the technical

Lma 2.5.1 For every $z, w \in \mathbb{H}$,

$d(z,w) \ge |\ln(\operatorname{Im} w / \operatorname{Im} z)|$, w/ equality $\Leftrightarrow \operatorname{Re} z = \operatorname{Re} w$.

Pf Let $p, q \in \overline{\mathbb{R}}$ the endpts of $\overleftrightarrow{zw}$.

If $\operatorname{Re} z = \operatorname{Re} w$, choose $p = \operatorname{Re} z$, $q = \infty$ $\Rightarrow g(u) = i\frac{u-p}{z-p}$



$\Rightarrow g(w) = i\frac{w - \operatorname{Re} z}{z - \operatorname{Re} z} = i\frac{\operatorname{Im} w}{\operatorname{Im} z}$

$\operatorname{Re} z \neq \operatorname{Re} w$: If $\operatorname{Im} z = \operatorname{Im} w$ the inequality is triv.

Con om

$\Rightarrow$ assume $\underbrace{\operatorname{Im} w > \operatorname{Im} z}_{(*)}$ (use $d(z,w) = d(w,z)$)

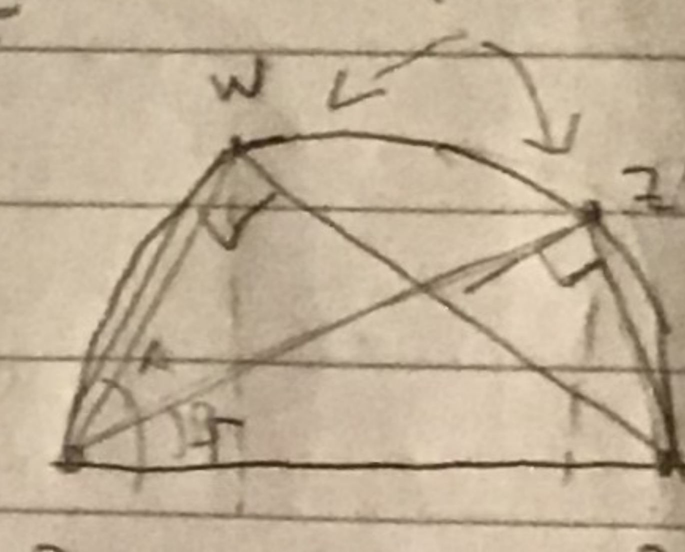
Now $g(u) := \dfrac{u-q}{u-p}\,\dfrac{z-p}{z-q}$ $\Rightarrow$

$$d(z,w) = d(w,z) = \left|\ln\left(\left|\frac{w-q}{w-p}\,\frac{z-p}{z-q}\right|\right)\right| = \left|\ln\left(\frac{|w-q|}{|w-p|} \Big/ \frac{|z-q|}{|z-p|}\right)\right|$$

w.l.o.g. $\underbrace{p * w * z * q}_{(**)}$ $\left( \dfrac{|w-p|}{|w-q|} \Big/ \dfrac{|z-p|}{|z-q|} = \dfrac{1}{\frac{|w-q|}{|w-p|} \Big/ \frac{|z-q|}{|z-p|}} \right)$

Using triangles (see book) we get

Thale's thm




$$\frac{|w-q|}{|w-p|} \Big/ \frac{|z-q|}{|z-p|} = \underbrace{\frac{\operatorname{Im} w}{\operatorname{Im} z}}_{* \Rightarrow\ > 1} \cdot \underbrace{\frac{|\operatorname{Re} z - p|}{|\operatorname{Re} w - p|}}_{(**) \Rightarrow\ > 1}$$

$|w-q|/|w-p| = \tan A = \dfrac{\operatorname{Im} w}{|\operatorname{Re} w - p|}$

$\Rightarrow d(z,w) = \ln\left(\dfrac{|w-q|}{|w-p|} \Big/ \dfrac{|z-q|}{|z-p|}\right) > \ln\left(\dfrac{\operatorname{Im} w}{\operatorname{Im} z}\right)$. $\square$

$\dfrac{|z-q|}{|z-p|} = \tan B = \dfrac{\operatorname{Im} z}{|\operatorname{Re} z - p|}$

Pf of (d3 + d4): (d5) $\Rightarrow$ assume $z = i$, $w = ti$, $t > 1$.

If $u$ a third pt, then

$$d(z,u) + d(u,w) \geq |\ln(\operatorname{Im} u / \operatorname{Im} z)| + |\ln(\operatorname{Im} w / \operatorname{Im} u)| \geq$$

$$|\ln(\operatorname{Im} u / \operatorname{Im} z) + \ln(\operatorname{Im} w / \operatorname{Im} u)| = |\ln(\operatorname{Im} w / \operatorname{Im} z)|$$

$= d(z,w)$ (since $z, w \in i\mathbb{R}$)

F⑧:3

Equality $\iff$ $\mathrm{Re}\, u = 0$ and $\ln(\mathrm{Im}\, u/\mathrm{Im}\, z) \cdot \ln(\mathrm{Im}\, w/\mathrm{Im}\, u) \geq 0$

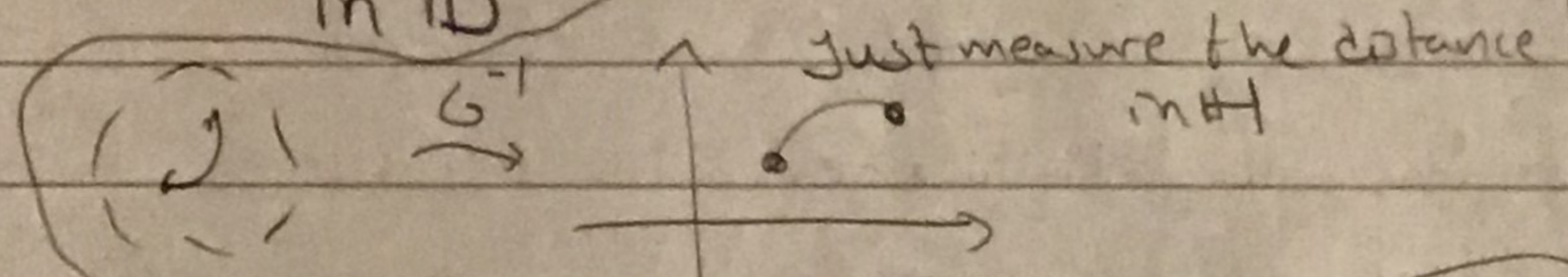
$\iff u \in i\mathbb{R}$, & $1 = \mathrm{Im}\, z \leq \mathrm{Im}\, u \leq \mathrm{Im}\, w$

$\iff u \in [z, w]$ $\square$

To understand the distance fcn better, go to

$\mathbb{D}$: $\underbrace{d_{\mathbb{D}}(z_1, z_2)}_{\text{define distance in } \mathbb{D}} = d_{\mathbb{H}}(G^{-1}(z_1), G^{-1}(z_2))$

distance in $\mathbb{H}$, $G: \mathbb{H} \to \mathbb{D}$ our usual identification




$\Rightarrow$ * $G, G^{-1}$ are <u>isometries</u> [preserves distance]

* (d1) - (d5) automatically hold for $d_{\mathbb{D}}$

[just map everything to the other model $\mathbb{H}$ for the pfs]

Q $f \in \mathrm{Möb}(\mathbb{D})$ isometry for $d_{\mathbb{D}}$, $f \in \mathrm{Möb}(\mathbb{H})$ isometry for $d_{\mathbb{H}}$.

Formula for $d_{\mathbb{D}}$

Assume $z_1, z_2 \in \mathbb{D}$ on the $\mathbb{D}$-line w/ endpts $p, q$

$\Rightarrow d_{\mathbb{D}}(z_1, z_2) = d_{\mathbb{H}}(G^{-1}(z_1), G^{-1}(z_2))$

and the map taking $G^{-1}(z_1) \mapsto i$, $G^{-1}(z_2) \mapsto i\mathbb{R}_{>0}$,

$G^{-1}(p) \mapsto 0$, $G^{-1}(q) \mapsto \infty$ is given by

$$g(w) := \frac{w - G^{-1}(p)}{w - G^{-1}(q)} \; \frac{G^{-1}(z_1) - G^{-1}(q)}{G^{-1}(z_1) - G^{-1}(p)} \quad \text{an FLT}$$

[assume for simplicity $G^{-1}(p), G^{-1}(q) \neq \infty$]

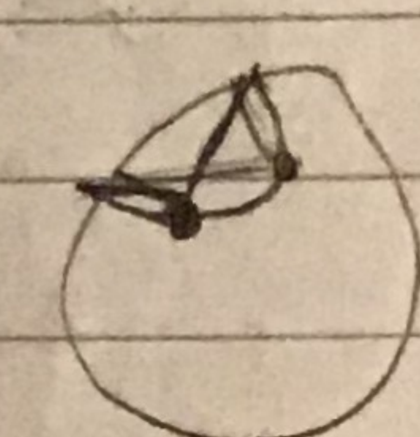
Let $\tilde{g}(w) = \frac{w - p}{w - q} \; \frac{z_1 - q}{z_1 - p}$, another FLT

Claim $\tilde{g} = g \circ G^{-1}$

Pf Enough to show

$$\tilde{g}(z_1) = g \circ G^{-1}(z_1) \quad (=1)$$
$$\tilde{g}(p) = g \circ G^{-1}(p) \quad (=0)$$
$$\tilde{g}(q) = g \circ G^{-1}(q) \quad = (=\infty)$$

$\Rightarrow$ result follows since FLTs uniquely det. by values in 3 pts.



$\Rightarrow g \circ G^{-1}(z_2) = \tilde{g}(z_2)$

$$\Rightarrow d_{\mathbb{D}}(z_1, z_2) = \left| \ln \frac{|z_2 - p|}{|z_2 - q|} \frac{|z_1 - q|}{|z_1 - p|} \right|$$

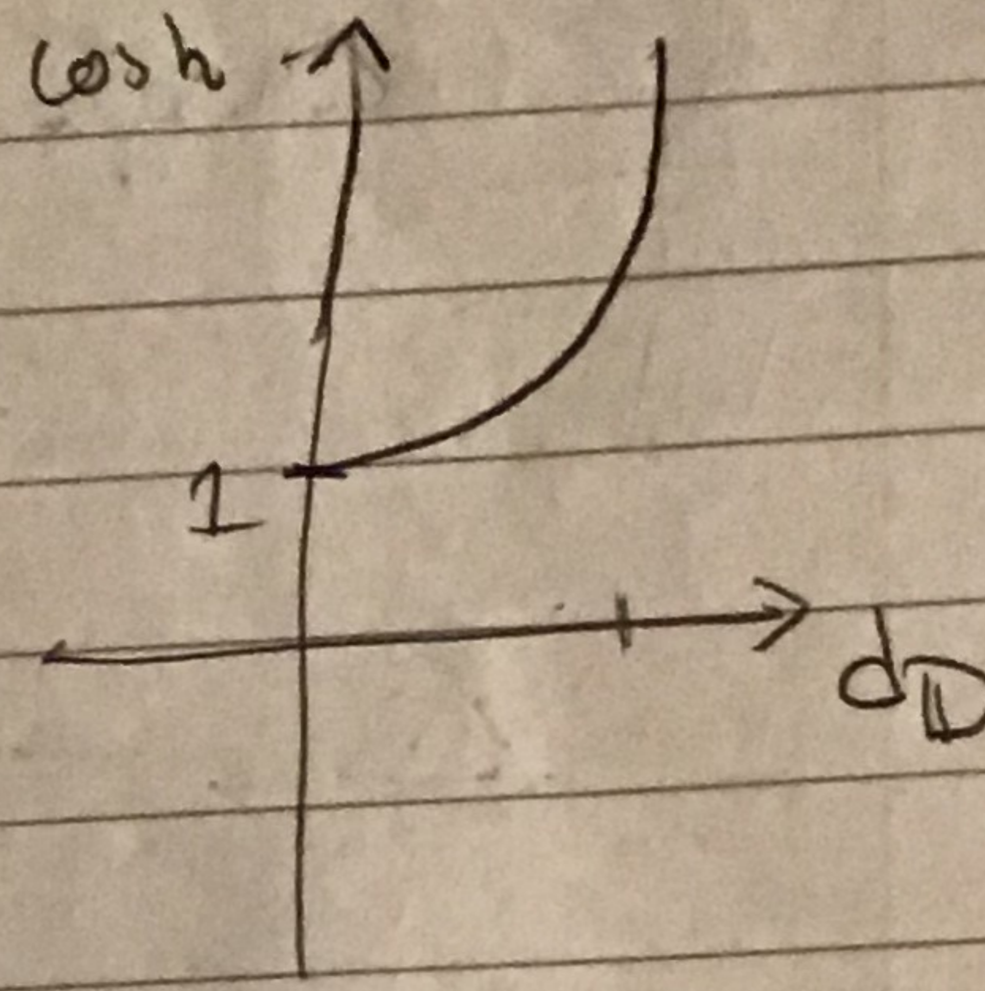
Now we want to remove the endpoints from the formula:

Prp $\cosh\left(d_{\mathbb{D}}(z_1, z_2)\right) = 1 + \frac{2|z_2 - z_1|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}$

$$\cosh\left(d_{\mathbb{H}}(z_1, z_2)\right) = 1 + \frac{|z_2 - z_1|^2}{2(\operatorname{Im} w_1)(\operatorname{Im} w_2)}$$

F⑧:4

Pf See book.

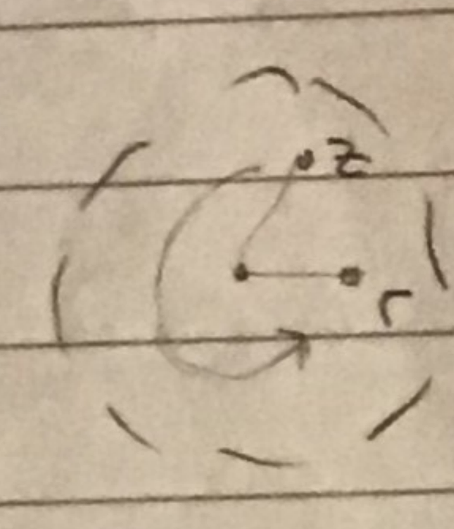



Recall $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\Rightarrow$ cosh invertible if we restrict to $x \geq 0$.

Geom ↑

To get a feeling for this




$d_{\mathbb{D}}(0, z) = d_{\mathbb{D}}(0, r)$ w/ $r = |z|$

[since {rotations around 0} $\subset$ Möb($\mathbb{D}$), f.ex. $g_\theta \in \text{Möb}^+(\mathbb{H}) \Leftrightarrow e^{i2\theta} z$,

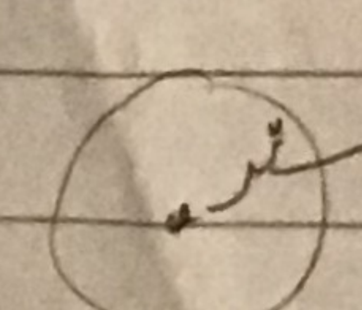
Q and elements in Möb ($\mathbb{D}$) preserve $d_{\mathbb{D}}$

($d_{\mathbb{H}}$ satisfies (d5) $\Rightarrow d_{\mathbb{D}}(f(z), f(w)) = d_{\mathbb{D}}(z, w)$ if $f \in \text{Möb}(\mathbb{D})$)

To compute $d_{\mathbb{D}}(0, r)$, note that $p = -1$, $q = 1$

$\Rightarrow d_{\mathbb{D}}(0, z) = \left| \ln\left(\left| \frac{0-(-1)}{0-1} \cdot \frac{r-1}{r-(-1)} \right|\right)\right| = \left| \ln \frac{1-r}{1+r} \right| = \ln \frac{1+r}{1-r}$

$\Rightarrow$ when $r \to 1$, this $\to \infty$



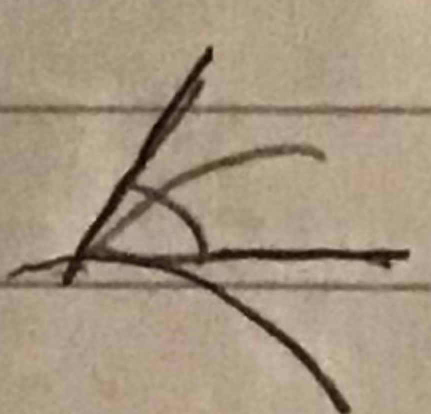
these 2 pts are very far away.

Def If $(X, d)$, $(Y, d')$ are metric spaces, then a homeomorphism

$f: X \to Y$ is an isometry if $d'(f(x_1), f(x_2)) = d(x_1, x_2)$ for all $x_1, x_2 \in X$.

## § 2.6 Angle measure. $\mathbb{H}$ as a conformal model

We can measure angles in $\mathbb{H}$ as we do in $\mathbb{E}^2$:



angle between 2 $\mathbb{H}$-lines intersecting at $p$

= euclidean angle between their tangent lines

This since:

Lma 2.6.1 Two angles A & B are congruent

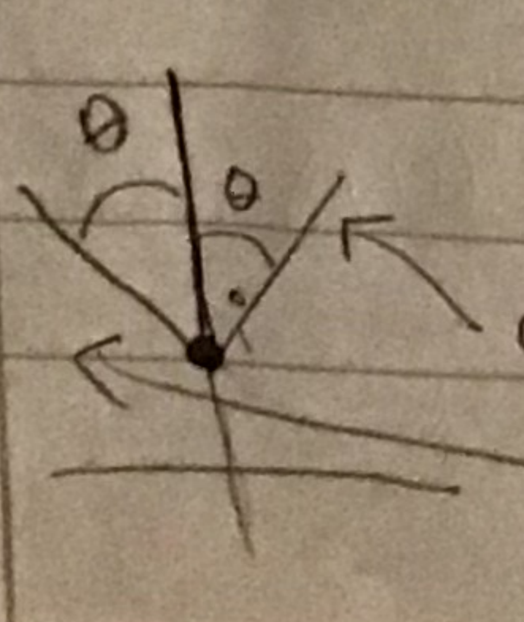
$\Longleftrightarrow$ they have the same euclidean measure.

Pf "$\Rightarrow$" Since FLT's and $z \mapsto \bar{z}$ preserve euclidean angles

"$\Leftarrow$" Let $A = \angle xyz$, $B = \angle uvw$ both with euclidean angle measure $= \theta$.

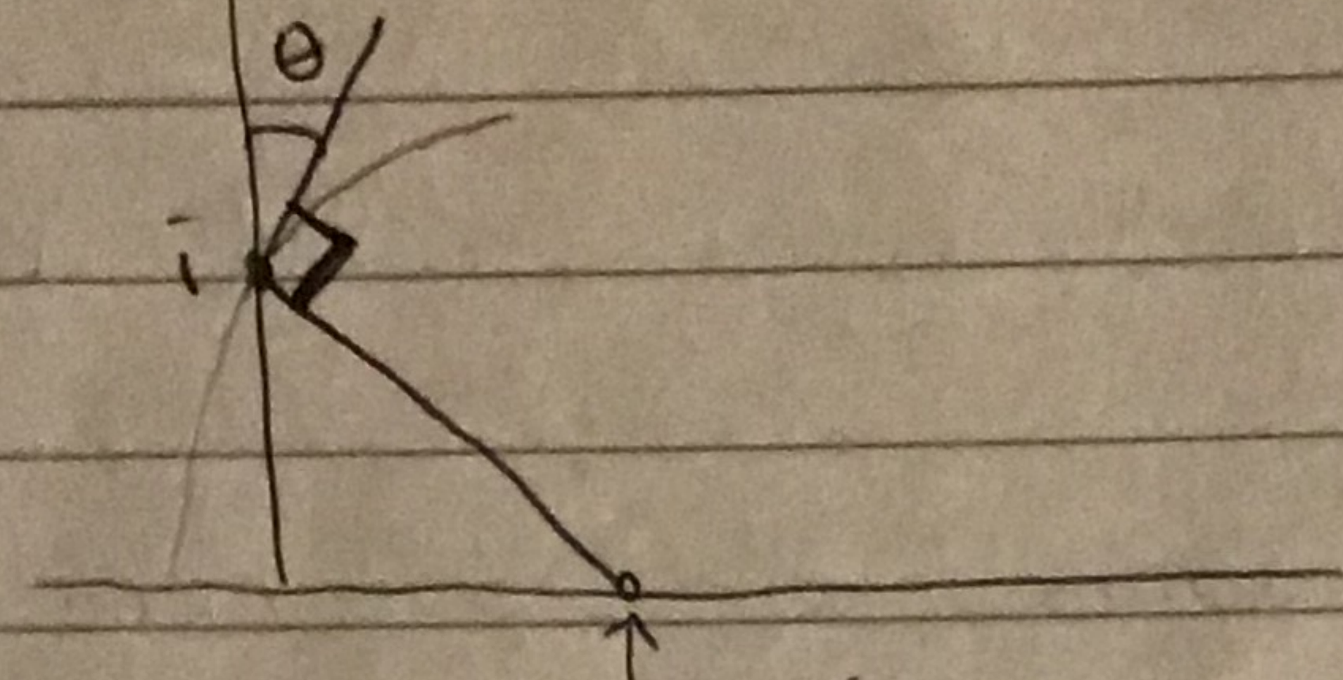
Axiom C4 $\Rightarrow$ might assume $\overrightarrow{yx} = \overrightarrow{vu} = [i, 0)$

Q Why? (Given angle & ray can find congruent angle containing that ray)



only one $\mathbb{H}$-line through $\cdot$ tangent to this

Q Why?




center of halfcircle, uniquely det by $\theta$ & side of $i\mathbb{R}$

From these 2 rays we get 2 angles. But they are congruent through the mapping $z \mapsto -\bar{z}$. $\square$

$\Rightarrow$ The Poincaré upper half-plane $\mathbb{H}$ is a <u>conformal</u> model for hyperbolic geometry

(meaning we can use the same measure of angles as in $\mathbb{E}^2$ in where we can see $\mathbb{H}$ as a subset).

Since $G : \mathbb{H} \to \mathbb{D}$ is an FLT, hence angle-preserving, the Poincaré disk model $\mathbb{D}$ is also a conformal model for hyperbolic geometry.