Today §2.8 Arc-length & area in $\mathbb{H}$ & $\mathbb{D}$

Last time $d_{\mathbb{H}}(z_1, z_2) = |\ln(|g(z_2)|)|$

where $g \in \text{Möb}^+(\mathbb{H})$ s.t. $g(\overleftrightarrow{z_1 z_2}) = i\mathbb{R}_{>0}$
$g(z_1) = i$

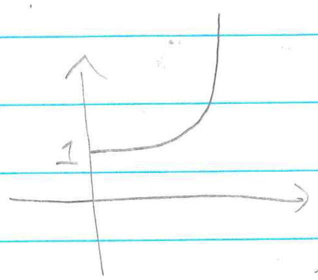
$d_{\mathbb{D}}(z_1, z_2) = d_{\mathbb{H}}(G^{-1}(z_1), G^{-1}(z_2))$

$\Rightarrow$ • G, G^{-1}, $f \in \text{Möb}(\mathbb{H})$, $h \in \text{Möb}(\mathbb{D})$ isometries.
• $d_{\mathbb{H}}, d_{\mathbb{D}}$ metrics satisfying d1)-d5)

"Explicit" formula: $\cosh(d_{\mathbb{H}}(z_1, z_2)) = 1 + \dfrac{|z_2 - z_1|^2}{2(\operatorname{Im} z_1)(\operatorname{Im} z_2)}$

$\cosh(d_{\mathbb{D}}(z_1, z_2)) = 1 + \dfrac{2|z_2 - z_1|^2}{(1-|z_1|^2)(1-|z_2|^2)}$

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$



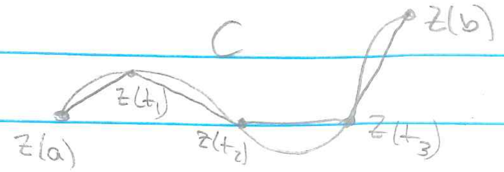
$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Preparation for next lecture: Exercise 2.7.8
Derive ~~these~~ formulas
involving cosh, sinh, tanh

Arc-length

Recall Let $C \subset (X, d)$ curve in a metric space parametrized by $z(t)$, $t \in [a,b]$. (assume injective)



Then the arc-length of C is

$$s(C) = \sup_{a = t_0 < t_1 < \dots < t_n = b} \sum_{i=1}^{n} d(z(t_i), z(t_{i-1}))$$

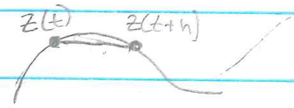
supremum over all possible finite partitions of $[a,b]$

(provided this is finite)

Note. finer partition makes $\sum d(,)$ larger (most often)

Ex $C \subset \mathbb{R}^2$ continously differentiable (C^1), param. by $z(t) = (x(t), y(t))$
$z' \neq 0$,

$$\Rightarrow \quad s(C) = \int_a^b \sigma(t)\, dt,$$



$$\sigma(t) = \lim_{h \to 0} \frac{d(z(t+h), z(t))}{|h|}$$

$$= \lim_{h \to 0} \frac{|(x(t+h), y(t+h)) - (x(t), y(t))|}{|h|}$$

$$= |(x'(t), y'(t))| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Notation $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$, or $ds^2 = dx^2 + dy^2$

the line element in $\mathbb{R}^2$

In $\mathbb{H}$ / $\mathbb{D}$

Lma (Ex 3.8.1)

Let $z_1, z_2 \in \mathbb{H}$. Then the shortest curve between them is the segment $[z_1, z_2]$.

pf Let $z(t)$, $a \le t \le b$ be a parametrization of a curve C w/ endpoints $z(a) = z_1$, $z(b) = z_2$, let $a = t_0 < t_1 < \dots < t_n = b$ be a partition

d3) + d4) $\Rightarrow$ $d_{\mathbb{H}}(z(t_{i+2}), z(t_i)) \le d_{\mathbb{H}}(z(t_{i+2}), z(t_{i+1}))$
$+ d_{\mathbb{H}}(z(t_{i+1}), z(t_i))$

w/ equality $\Leftrightarrow$ $z(t_{i+1}) \in [z(t_i), z(t_{i+2})]$

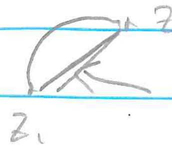
$\Rightarrow$ $d_{\mathbb{H}}(z_1, z_2) \le \sum_i d_{\mathbb{H}}(z(t_i), z(t_{i-1})) \le \sup_i \sum d_{\mathbb{H}}(z(t_i), z(t_{i-1}))$
$= s(C)$

w/ equality $\Leftrightarrow$ $z(t)$ parametrizes the segment $[z_1, z_2]$. □

$\Rightarrow$

To compute the arc-length of $[z_1, z_2]$ we can use $d_{\mathbb{H}}(z_1, z_2)$.

To compute the arc-length of more general curves,



such as this one, we want to use

$$s(t) = \int_a^t \sigma(t)\,dt \quad \text{as in } \mathbb{R}^2,$$

$\boxed{x}$

$\Rightarrow$ must derive an expression for the line element

$$ds = \sigma(t)\,dt.$$

Proceed similarly as in $\mathbb{R}^2$, that is, find an explicit expression for

$$\sigma(t) = \lim_{h \to 0} \frac{(d_{\mathbb{H}}(z(t+h), z(t))}{|h|}$$

Use $\bullet \cosh\Big(\underbrace{d_{\mathbb{H}}(z(t+h), z(t))}_{d(h)}\Big) = 1 + \dfrac{|z(t+h) - z(t)|^2}{2(\operatorname{Im} z(t+h))\cdot(\operatorname{Im} z(t))}$

$\bullet$ Taylor's formula for $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$:

$$1 + \frac{x^2}{2} + \eta(x)x^2 \quad \text{where} \quad \lim_{x \to 0} \eta(x) = 0$$

$$\Rightarrow \quad 1 + \frac{(d(h))^2}{\cancel{2}} + \eta(d(h))(d(h))^2 = 1 + \frac{|z(t+h) - z(t)|^2}{\cancel{2}(\operatorname{Im} z(t+h))(\operatorname{Im} z(t))}$$

$$\Longrightarrow \quad \frac{(d(h))^2}{|h|^2}\Big(1 + \eta(d(h))\Big) = \frac{|z(t+h) - z(t)|^2}{|h|^2} \, \frac{1}{(\operatorname{Im} z(t+h))(\operatorname{Im} z(t))}$$

divide by $|h|^2$

F(9):3

LHS $\xrightarrow[|h|\to 0]{}$ $\lim_{|h|\to 0} \frac{d_{\mathbb{H}}(z(t+h), z(t))^2}{|h|^2} \cdot (1+0) = (\sigma(t))^2$

RHS $\xrightarrow[|h|\to 0]{}$ $\frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}{(\operatorname{Im}(z(t))^2}$ (sam as in $\mathbb{R}^2$, assuming $z \in C^1$, $z'(t) \neq 0 \ \forall t$)

$$\Rightarrow \boxed{ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \text{line-element in } \mathbb{H}}$$

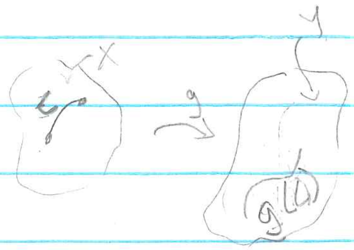
$$\Rightarrow s(t) = \int_a^t \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y}\, dt \quad [\text{since } \sigma(t) > 0]$$

<u>In $\mathbb{D}$</u>

<u>Oblig</u> $ds^2 = 4 \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$ (perform similar steps as above)

<u>or</u> use

<u>Lma</u> Assume $g : (X, d_X) \to (Y, d_Y)$ isometry, and let $z(t)$, $t \in [a,b]$ be a curve in X. Then the arc-length of the curve $g \circ z(t) \subset (Y, d_Y)$ equals the arc-length of $z(t) \subset (X, d_X)$



<u>Pf</u> Follows since

$d_Y(g \circ z(t_i), g \circ z(t_{i-1})) = d_X(z(t_i), z(t_{i-1})) \quad \forall i \quad \square$

Ex 2.8.3 Compute the arc-length of the hyperbolic circle C w/ center $m \in \mathbb{H}$ & hyperbolic radius ς.

Sol ① May assume $C \subset \mathbb{D}$ w/ center in 0

If $C \subset \mathbb{H}$, map it to $\mathbb{D}$ w/ the isometry G

If center at $p \in \mathbb{D}$, apply $\frac{z-p}{-\bar{p}z+1} \in \text{Möb}^+(\mathbb{D})$

$[\alpha=1, \beta=-p \Rightarrow \alpha\bar{\alpha}-\beta\bar{\beta}=1-|p|^2>0]$

Lecture 8 $\Rightarrow$ the euclidean radius r of C satisfies

$$\varsigma = \ln\frac{r+1}{r-1} \iff r = \tanh(\varsigma/2)$$

$\Rightarrow$ C may be parametrized as $z(t) = re^{it} = (r\cos t, r\sin t)$, $t \in [0, 2\pi]$

$\Rightarrow \frac{dx}{dt} = -r\sin t$, $\frac{dy}{dt} = r\cos t$, $x^2+y^2=r^2 \Rightarrow$

$$\left(\frac{ds}{dt}\right)^2 = 4\,\frac{(-r\sin t)^2+(r\cos t)^2}{(1-r^2)^2} = \frac{4r^2}{(1-r^2)^2}$$

$$\Rightarrow s(C) = \int_0^{2\pi} \frac{2r}{(1-r^2)}\,dt = \frac{4\pi r}{1-r^2} = \pi \boxed{\frac{4\tanh\left(\frac{\varsigma}{2}\right)}{1-\tanh^2\left(\frac{\varsigma}{2}\right)}} = 2\pi\sinh(\varsigma) =$$

$= 2\sinh(\varsigma)$ by Exc. 2.7.8

F(9):4

$$\pi(e^{\rho}-e^{-\rho}) = \pi\left(2\rho + \frac{\rho^3}{3} + \frac{\rho^5}{60} + \dots\right)$$

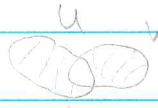
$\Rightarrow$ if ρ small this is $\approx 2\pi\rho$ = circumference of the euclidean circle of radius ρ

But as $\rho \to \infty$, this is $\approx \pi e^{\rho}$ which is very large.

Area in $\mathbb{H}$

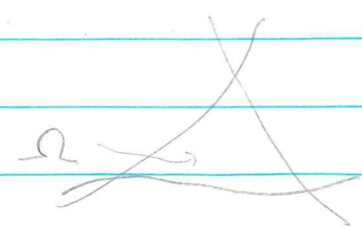
A reasonable area fcn should satisfy

- $A(U \cup V) + A(U \cap V) = A(U) + A(V)$



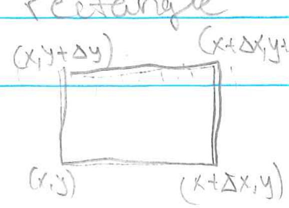
- $A(\{pt\}) = A(\text{curve}) = 0$
- Congruent sets should have the same area
  ($\Leftrightarrow$ Möbius transformations should be area-preserving)

Rmk We will only measure area of regions bdd by a finite nbr of C^1-curves.



Idea Cover Ω by rectangles

$R(\Delta x, \Delta y)$ = euclidean rectangle w/ vertices



and $A_{\mathbb{H}}(\Omega) \approx \sum A_{\mathbb{H}}(R(\Delta x, \Delta y))$

$$\longrightarrow \iint_{\Omega} \lim_{\substack{\Delta x, \\ \Delta y \to 0}} \frac{A_{\mathbb{H}}(R(\Delta x, \Delta y))}{\Delta x \, \Delta y} \, dx\,dy$$

when we let the rectangles be smaller and smaller.

<u>Reasonable</u> to require $A_{\mathbb{H}}(R(\Delta x, \Delta y)) \approx d_{\mathbb{H}}(x, x+\Delta x) \cdot d_{\mathbb{H}}(y, y+\Delta y)$, as $\Delta x, \Delta y \to 0$

since this is a straightforward generalization of the euclidean case

And from the formula for arc-length, applied to $z_x(t) = x + t$, $0 \leq t \leq 1$, $z_y(t) = i(y + t)$,

$\Delta x, \Delta y > 0$

we see $\lim_{\Delta x \to 0} \frac{d_{\mathbb{H}}(x, x+\Delta x)}{|\Delta x|} = \frac{1}{y} = \lim_{\Delta y \to 0} \frac{d_{\mathbb{H}}(y, y+\Delta y)}{|\Delta y|}$

$\left[\text{since } \lim_{h \to 0} \frac{d_{\mathbb{H}}(z(t+h), z(t))}{|h|} = \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y} \right]$

$$\Rightarrow \boxed{A_{\mathbb{H}}(\Omega) = \iint_{\Omega} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \overbrace{\frac{d_{\mathbb{H}}(x, x+\Delta x)\, d_{\mathbb{H}}(y, y+\Delta y)}{|\Delta x||\Delta y|}}^{A_{\mathbb{H}}(R(\Delta x, \Delta y))} \, dx\,dy = \iint_{\Omega} \frac{1}{y^2} \, dx\,dy}$$

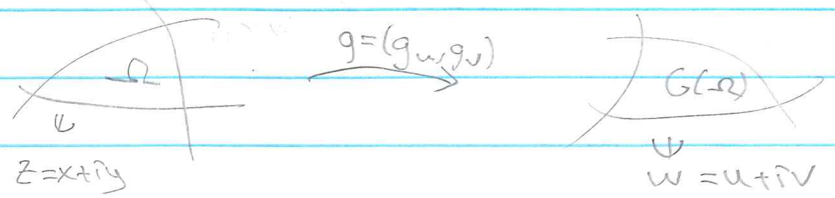
<u>Oblig</u>: Make a similar derivation for $A_{\mathbb{D}}$.

<u>Prp</u> $A_{\mathbb{H}}$ is invariant under congruence.

<u>Pf</u> Must prove $A_{\mathbb{H}}(\Omega) = A_{\mathbb{H}}(g\,\Omega)$ if $g \in \text{Möb}(\mathbb{H})$

i) Assume $g \in \text{Möb}^+(\mathbb{H}) \Rightarrow g(z) = \dfrac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{R}$

$ad - bc = 1$



<u>Recall</u> (several variable calculus)

$$A_{\mathbb{H}}(\Omega') = \iint_{\Omega' = g(\Omega)} \frac{du\,dv}{v^2} = \iint_{\Omega} |J(g)(z)| \frac{dx\,dy}{|\mathrm{Im}\,g(z)|^2}$$

$$= \det \begin{bmatrix} \frac{\partial g_u}{\partial x} & \frac{\partial g_u}{\partial y} \\ \frac{\partial g_v}{\partial x} & \frac{\partial g_v}{\partial y} \end{bmatrix} = \left(\frac{\partial g_u}{\partial x}\right)^2 + \left(\frac{\partial g_v}{\partial x}\right)^2 =$$

g analytic
use Cauchy-Riemann eqns

$= |g'(z)|^2$, and in our case

$$g'(z) = \frac{ad-bc}{(cz+d)^2} = \frac{1}{(cz+d)^2} \Rightarrow |J(g)(z)| = \frac{1}{|cz+d|^4}$$

Also $\operatorname{Im}(g(z)) = \dfrac{y}{|cz+d|^2}$

Q Check this $\left(\operatorname{Im} \dfrac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \operatorname{Im} \dfrac{adz + bc\bar{z}}{|cz+d|^2} = \dfrac{y(ad-bc)}{|cz+d|^2} \right)$

$$\Rightarrow A_{\mathbb{H}}(\Omega') = \iint_{\Omega} \frac{1}{|cz+d|^4} \frac{dx\,dy}{\left(\frac{y}{|cz+d|^2}\right)^2} = \iint_{\Omega} \frac{dx\,dy}{y^2} = A_{\mathbb{H}}(\Omega)$$

ii) $g(z) = -\bar{z}$: then g preserves the y-coord &

has Jacobian $= -1$

$\Rightarrow A_{\mathbb{H}}(\Omega') = A_{\mathbb{H}}(\Omega)$ also in this case

The result follows since $\text{Möb}(\mathbb{H})$ is generated by

$\text{Möb}^+(\mathbb{H})$ & $z \mapsto -\bar{z}$. $\square$