PSS 4 : 1 – Hints

5.3.1 Ans $x(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$

$$E(u,v) = \langle x_u, x_u \rangle = (f'(u))^2 + (g'(u))^2$$

$$F(u,v) = \langle x_u, x_v \rangle = 0$$

$$G(u,v) = \langle x_v, x_v \rangle = (f(u))^2$$

5.4.5 b) $y(u,v) = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, v - \tanh v \right)$

surface of rotation w/ $f(v) = \frac{1}{\cosh v}$, $g(v) = v - \tanh v$

(using the notation above)

$\Rightarrow$ regular if $(f'(v), g'(v)) \neq (0,0)$ $\forall \underbrace{v \in (0,\infty)}_{\text{which is our domain of definition, given in a)}}$

Since $f'(v) = \frac{\sinh v}{\cosh^2 v}$, $g'(v) = \tanh^2 v$

and these are $0 \Leftrightarrow v=0$, y gives a regular param.

To compute the metric you can use 5.3.1

$\Rightarrow E(u,v) = (f(v))^2 = \frac{1}{\cosh^2 v}$, $F(u,v) = 0$

$$G(u,v) = (f'(v))^2 + (g'(v))^2 = \frac{\sinh^2(v)(1+\sinh^2 v)}{\cosh^4 v} = \tanh^2 v.$$

5.4.6 $\langle\, , \rangle^\lambda$ metric: Let $E_\lambda = \langle x_u, x_u \rangle^\lambda$

$F_\lambda = \langle x_u, x_v \rangle^\lambda$

$G_\lambda = \langle x_v, x_v \rangle^\lambda$

$\Rightarrow E_\lambda = \lambda E_1$, $F_\lambda = \lambda F_1$, $G_\lambda = \lambda G_1$

$\Rightarrow$ these are smooth fcns. And clearly $\langle\, , \rangle^\lambda_p$

defines an inner prod on T_pS $\forall p$

S_λ not isometric to $S_{\lambda'}$ if cpct: Compute their areas

$\mathbb{R}^2_\lambda$ is isometric to $\mathbb{R}^2_{\lambda'}$: Use $f_{\lambda/\lambda'}(p) = \frac{\lambda}{\lambda'} p$

5.4.7 $\quad$ g local diffeo $\Leftrightarrow$ dg_p isomorphism $\forall p$

$$\Leftrightarrow \underbrace{(dg_p(v)=0 \Rightarrow v=0)}$$

prove this holds if
g is conformal

g angle preserving $\quad$ If $v, w \in T_pS$ then the angle ϕ
between them is given by $\phi \in [0,\pi]$ that satisfies

$$\langle v, w\rangle = \cos\phi \,\|v\|\,\|w\|$$

Prove that the angle between $dg_p(v)$ & $dg_p(w)$
is given by ϕ.

g^{-1} conformal if g diffeo $\quad$ use that $d(g^{-1})_{g(p)} = (dg_p)^{-1}$.

[The "new" k is given by $\frac{1}{k\circ g^{-1}}$]

S_λ & $S_{\lambda'}$ conformally equivalent
Use $g = \text{identity}$, $k = \frac{\lambda}{\lambda'}$ (or $\frac{\lambda'}{\lambda}$)

5.5.1 $\quad$ By Prp 5.5.3 we have $K = \dfrac{eg-f^2}{EG-F^2}$

Compute : $\quad EG-F^2 = (g'^2+h'^2)g^2$

$$N = \frac{r_u \times r_v}{\|r_u\times r_v\|} = \frac{(-h'g\cos v, -h'g\sin v, g'g)}{\sqrt{EG-F^2}}$$

$$e = N\cdot r_{uu} = \frac{-h'gg''+h''g'g}{\sqrt{EG-F^2}}, \quad f=0, \quad g = \frac{g^2h'}{\sqrt{EG-F^2}} \quad \text{(bad notation)}$$

$$\Rightarrow eg-f^2 = \frac{g^3h'(h''g'-h'g'')}{EG-F^2} \quad \text{\& the result should now follow}$$

Such a surface has $K\equiv 0$ if h constant or $h = cg+d$, $c, d \in \mathbb{R}$

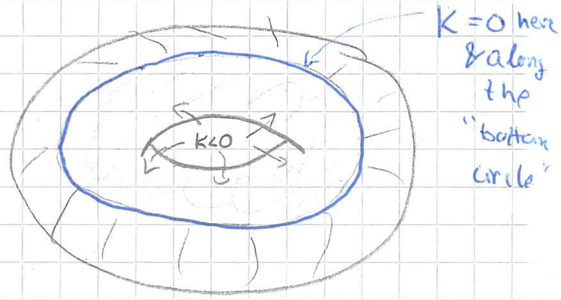
PSS④:2 - Hints

5.5.2 Use 5.5.1 to get $K = \dfrac{\cos u}{a(a\cos u + b)}$

$\Rightarrow K < 0$ if $u \in (\pi/2, 3\pi/2)$

$K > 0$ if $u \in (-\pi/2, \pi/2)$

$K = 0$ if $u = \frac{\pi}{2}, \frac{3\pi}{2}$



$K>0$ on the "outside"
$K<0$ on the "inside"

5.5.5 What happens to the Gaussian curvature when we scale the metric as in exc 5.4.6?

Ans It is scaled by $\frac{1}{\lambda}$:

[Can be seen from e.g. Brioschi's formula
$(EG-F^2)^2$ scaled by λ^4, the determinants
of the 3x3-matrices are scaled by λ^3]

5.5.10 $S \subset \mathbb{R}^3$ compact by assumption

$H = k_1 + k_2 = 0 \Rightarrow k_1 = -k_2$

But $K = k_1 k_2 = -k_2^2$.

Now use Prp 5.5.6.

5.6.2 Compute $\frac{d}{dt}\beta(h(t))$, $\frac{d^2}{dt^2}\beta(h(t))$ & $D\frac{d^2}{dt^2}\beta(h(t))$

& compare w. $\beta'(t)$, $D\beta''(t)$.

5.6.6 One suggestion: $\alpha_m(t) = (\cos 2mt, \sin 2mt, t/2\pi)$,

$t \in [0, 2\pi]$, $m \in \mathbb{Z}$.

5.6.11 You can argue that the Christoffel symbols don't change under the rescaling
$\Rightarrow D\beta'(t)$ doesn't change.
And clearly $\beta'(t)$ is independent of metric.

For the second statement you can use the results from Exercise 5.4.6.

5.7.2 The geodesic polar coord for the scaled metric is given by the param

$$x^\lambda(r, \theta) = \exp_p\left(\tfrac{r}{c}(\cos\theta, \sin\theta)\right)$$

where $\exp_p$ is the exponential map for the un-scaled metric.

The reason is that the geodesic
$r \mapsto \exp_p(r(\cos\theta, \sin\theta))$, $r \in [0, r_0]$ has length cr_0 w.r.t. $\|\cdot\|_\lambda$.

$$\Rightarrow G_\lambda(r,\theta) = \langle x^\lambda_\theta(r,\theta), x^\lambda_\theta(r,\theta)\rangle^\lambda = \lambda \langle x_\theta(\tfrac{r}{c},\theta), x_\theta(\tfrac{r}{c},\theta)\rangle$$

$$= c^2 G(\tfrac{r}{c}, \theta)$$

$F_\lambda(r,\theta) = 0$ by Gauss' lma, $E_\lambda(r,\theta) = 1$ since we parametrize the geodesics by arc-length.

PSS 4 : 3 - Hints

5.9.5 Use Gauss-Bonnets thm

$$\iint_S K\,dA = 2\pi\,\chi(S)$$

+ Thm 3.1.14

5.9.6 Let $A(S_i)$ be the area of S_i, $i = 1, 2, \emptyset$

Gauss-Bonnet $\Rightarrow$ $\underbrace{K A(S_i)}_{\text{use } S_1 \cap S_2 = \partial S_i \text{ geodesic}} = 2\pi\,\chi(S_i)$

$\Rightarrow A(S_1)/A(S_2) = \chi(S_1)/\chi(S_2)$

If $K > 0$ then $S \approx S^2$ by Exc 5.9.5

$\rightarrow \chi(S) = 2$ by Thm 3.1.14

and since $K A(S) = 2\pi \cdot \chi(S) = 4\pi$

$$\| $$

$$\underbrace{K A(S_1)}_{>0} + \underbrace{K A(S_2)}_{>0}$$

it follows that $\chi(S_1) = \chi(S_2) = 1$

5.9.7 By assumption $K = -1$

$\Rightarrow$ $A(S) = 8\pi$ by Gauss-Bonnet