MAT4510, Fall 2023
Solutions to assignment.

Problem 1. Let

$$S := \{(x, y, z) \in \mathbb{R}^3 \,|\, xyz = 1 \text{ and } x, y, z > 0\}.$$

(i) Show that S is a regular surface.

(ii) Calculate the Gauss curvature of S.

Solution: (i) Let $h : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = xyz.$$

The gradient of f is

$$\nabla f(x, y, z) = (yz, xz, xy).$$

Because $\nabla f \neq 0$ at every point in $f^{-1}(1)$, it follows that $S = f^{-1}(1)$ is a regular surface.

(ii) The gradient of f is normal to S at every point on S, hence

$$N(x, y, z) := \frac{(yz, xz, xy)}{\sqrt{y^2z^2 + x^2z^2 + x^2y^2}}$$

is a smooth unit normal field on S. We obtain a global parametrization of S by regarding it as the graph of a function. Namely, let

$$U := \{(u, v) \in \mathbb{R}^2 \,|\, u, v > 0\}$$

and define $F : U \to S$ by

$$F(u, v) := (u, v, \frac{1}{uv}).$$

We now compute the corresponding matrix of the first fundamental form.

$$\partial_1 F = (1, 0, -\frac{1}{u^2v}), \quad \partial_2 F = (0, 1, -\frac{1}{uv^2}).$$

This gives

$$\begin{aligned} g_{11} &= \|\partial_1 F\|^2 = 1 + \frac{1}{u^4 v^2} \\ g_{22} &= \|\partial_2 F\|^2 = 1 + \frac{1}{u^2 v^4} \\ g_{12} &= \langle \partial_1 F, \partial_2 F \rangle = \frac{1}{u^3 v^3}. \end{aligned}$$

Therefore,

$$\det G = g_{11} g_{22} - g_{12}^2 = 1 + \frac{1}{u^4 v^2} + \frac{1}{u^2 v^4}.$$

Next, we compute the matrix of the second fundamental form.

$$\begin{aligned} \partial_1^2 F &= (0, 0, \frac{2}{u^3 v}), \\ \partial_2^2 F &= (0, 0, \frac{2}{u v^3}), \\ \partial_1 \partial_2 F &= (0, 0, \frac{1}{u^2 v^2}). \end{aligned}$$

Let

$$\tilde{N}(u, v) := N(F(u, v)) = \frac{(v, u, u^2 v^2)}{\sqrt{u^2 + v^2 + u^4 v^4}}.$$

Then $h_{ij} = \langle \partial_i \partial_j F, \tilde{N} \rangle$ is given by

$$\begin{aligned} h_{11} &= \frac{2v}{u\sqrt{u^2 + v^2 + u^4 v^4}}, \\ h_{22} &= \frac{2u}{v\sqrt{u^2 + v^2 + u^4 v^4}}, \\ h_{12} &= \frac{1}{\sqrt{u^2 + v^2 + u^4 v^4}}. \end{aligned}$$

This yields

$$\det H = \frac{3}{u^2 + v^2 + u^4 v^4},$$

so the Gauss curvature K on S is

$$K(x, y, z) = K(F(x, y)) = \frac{\det H(x, y)}{\det G(x, y)} = \frac{3x^4 y^4}{(x^2 + y^2 + x^4 y^4)^2}.$$

Problem 2.

Let a, b be real numbers with $0 < b < a$, and let S be the surface of revolution obtained by revolving the circle

$$\{(x, 0, z) \in \mathbb{R}^3 \mid (x - a)^2 + z^2 = b^2\}$$

about the z–axis. Describe the region in S where the Gauss curvature is positive and the region where it is negative.

Solution: We parametrize the circle C by $c : \mathbb{R} \to \mathbb{R}^3$,

$$c(t) = (r(t), 0, h(t)),$$

where

$$r(t) = a + b\cos(t/b), \quad h(t) = b\sin(t/b).$$

For any open interval I of lenght at most $2b\pi$ and any open interval J of length at most 2π the surface S has a local parametrization

$$F : I \times J \to S, \quad (t, \theta) \mapsto (r(t)\cos\theta, r(t)\sin\theta, h(t)).$$

Since the curve c has unit speed, we can apply the results of Problem 4 for the 20th September, according to which the Gauss curvature K of S satisfies

$$K(F(t, \theta)) = -\frac{\ddot{r}(t)}{r(t)} = \frac{\cos(t/b)}{b(a + b\cos(t/b))},$$

which has the same sign as $\cos(t/b)$. This means the the region in S obtained by revolving the (open) right semicircle of C has positive curvature, whereas the region obtained by revolving the left semicircle has negative curvature. In other words, if $(x, y, z) \in S$ then $K(x, y, z)$ has the same sign as $x^2 + y^2 - a^2$.

Problem 3.

Let S be the surface of revolution obtained by revolving a curve $\gamma : I \to \mathbb{R}^3$ about the z–axis, where γ has the form

$$\gamma(t) = (r(t), 0, h(t))$$

and $r > 0$. Recall that S has a local parametrization $F : I \times J \to S$ given by

$$F(t, \phi) = (r(t)\cos\phi, r(t)\sin\phi, h(t))$$

for any open interval J of length 2π.

(i) For fixed $\phi \in J$ the curve $c : I \to S$ given by

$$c(t) := F(t, \phi)$$

is called a *line of longitude*. Show that c is a geodesic if and only if γ has constant speed.

(ii) For fixed $t \in I$ the curve $c : J \to S$ given by

$$c(\phi) := F(t, \phi)$$

is called a *line of latitude*. Show that c is a geodesic if and only if $\dot{r}(t) = 0$.

Solution: (i) The curve c is a geodesic if and only if $\ddot{c}(t) = \partial_1^2 F(t, \phi)$ is orthogonal to the tangent space $T_{c(t)}S$ for all $t \in I$. Since the vectors $\partial_1 F(t, \phi), \partial_2 F(t, \phi)$ form a basis for $T_{F(t,\phi)}S$, it follows that c is a geodesic if and only if

$$\langle \partial_1^2 F(t, \phi), \partial_i F(t, \phi) \rangle = 0$$

for $i = 1, 2$ and all $t \in I$. We calculate

$$\begin{aligned}
\partial_1 F(t, \phi) &= (\dot{r}(t) \cos \phi, \dot{r}(t) \sin \phi, \dot{h}(t)), \\
\partial_2 F(t, \phi) &= (-r(t) \sin \phi, r(t) \cos \phi, 0), \\
\partial_1^2 F(t, \phi) &= (\ddot{r}(t) \cos \phi, \ddot{r}(t) \sin \phi, \ddot{h}(t)).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\langle \partial_1^2 F(t, \phi), \partial_1 F(t, \phi) \rangle &= \dot{r}(t)\ddot{r}(t) + \dot{h}(t)\ddot{h}(t) = \frac{1}{2} \frac{d}{dt} \|\dot{\gamma}(t)\|^2, \\
\langle \partial_1^2 F(t, \phi), \partial_2 F(t, \phi) \rangle &= 0.
\end{aligned}$$

Therefore, c is a geodesic if and only if γ has constant speed.

(ii) We have

$$\ddot{c}(t) = \partial_2^2 F(t, \phi) = (-r(t) \cos \phi, -r(t) \sin \phi, 0),$$

and

$$\begin{aligned}
\langle \partial_2^2 F(t, \phi), \partial_1 F(t, \phi) \rangle &= -r(t)\dot{r}(t), \\
\langle \partial_2^2 F(t, \phi), \partial_2 F(t, \phi) \rangle &= 0.
\end{aligned}$$

Since $r > 0$, the curve c is a geodesic if and only if $\dot{r}(t) = 0$.

Problem 4.

Let S be a regular surface. A 1–*form* on S is a rule α that assigns to every point $p \in S$ a linear map $\alpha_p : T_pS \to \mathbb{R}$. A 1–form α is called *smooth* if for every smooth vector field X on S the map

$$\alpha(X) : S \to \mathbb{R}, \quad p \mapsto \alpha_p(X_p)$$

is smooth.

(i) Show that for any smooth 1–form α on S the map

$$d\alpha : \mathfrak{X}(S) \times \mathfrak{X}(S) \to C^\infty(S)$$

given by

$$d\alpha(X, Y) = \partial_X(\alpha(Y)) - \partial_Y(\alpha(X)) - \alpha([X, Y])$$

is $C^\infty(S)$–bilinear.

(ii) Deduce from (i) that for any $p \in S$ there is a unique skew-symmetric, bilinear map

$$(d\alpha)_p : T_pS \times T_pS \to \mathbb{R}$$

such that for any smooth vector fields X, Y defined in a neighbourhood of p in S one has

$$d\alpha(X, Y) = (d\alpha)_p(X_p, Y_p).$$

Solution: See the proof of Proposition 11.1 in the lecture notes.