Vector fields, the covariant derivative, and curvature

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November 22, 2023

Contents

1 Some definitions

If $U \subset \mathbb{R}^m$ is an open set and $h : U \to \mathbb{R}^n$ a smooth map then $\partial_i h : U \to \mathbb{R}^n$ will denote the ith partial derivative of h. In other words, if $(u^1, \dots, u^m)$ are the standard coordinates on $\mathbb{R}^m$ then

$$\partial_i h = \frac{\partial h}{\partial u^i}.$$

Let $S \subset \mathbb{R}^3$ be a regular surface and $h : S \to \mathbb{R}^n$ a smooth map. The **differential** of h at a point $p \in S$ is the unique linear map

$$d_p h : T_p S \to \mathbb{R}^n$$

such that for any smooth curve $\gamma : (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ one has

$$d_p h(\dot\gamma(0)) = \left.\frac{d}{dt}\right|_0 h(\gamma(t)).$$

A map $X : S \to \mathbb{R}^3$ is called a **vector field** if $X(p) \in T_p S$ for all $p \in S$. A map $N : S \to \mathbb{R}^3$ is called a **normal field** if $N(p) \perp T_p S$ for all $p \in S$. If in addition $\|N(p)\| = 1$ for all p then N is called a **unit normal field**. One often writes X_p instead of $X(p)$, and similarly for N.

For example, any local parametrization $F : U \to S$ gives rise to **coordinate vector fields** X_1, X_2 on $F(U)$ satisfying

$$X_i \circ F = \partial_i F.$$

Thus, if $u \in U$ and $p = F(u)$ then $X_i(p) = \partial_i F(u)$. Since $X_1(p), X_2(p)$ is a basis for $T_p S$ for every $p \in F(U)$, we also get a smooth normal field

$$N = \frac{X_1 \times X_2}{\|X_1 \times X_2\|}$$

on $F(U)$.

2 Gauss curvature

Let $S \subset \mathbb{R}^3$ be a regular surface. The **Gauss curvature** $K : S \to \mathbb{R}$ is defined as follows. Given $p \in S$, choose a smooth unit normal field N defined in a neighbourhood W of p in S. We now look at the differential of N as a map $W \to S^2$. Because

$$T_{N(p)}S^2 = N(p)^\perp = T_pS,$$

the differential

$$dN_p : T_pS \to T_{N(p)}S^2$$

is in fact an endomorphism of T_pS. The Gauss curvature at p is defined to be the determinant of this endomorphism, i.e.

$$K(p) = \det(dN_p).$$

Then $K(p)$ is independent of the choice of N, because

$$\det(d(-N)_p) = \det(-dN_p) = \det(dN_p).$$

The linear map

$$W_p = -dN_p : T_pS \to T_pS$$

is called the **Weingarten map**. Clearly,

$$\det(W_p) = \det(dN_p) = K(p).$$

The bilinear map

$$II_p : T_pS \times T_pS \to \mathbb{R}, \quad (u, v) \mapsto \langle W_p(u), v\rangle$$

is called the **second fundamental form**.

Our next goal is to describe the second fundamental form and the Gauss curvature in terms of a local parametrization $F : U \to S$, where $U \subset \mathbb{R}^2$ is an open set. Let N be a smooth unit normal field on $F(U)$. We now look at the second order partial derivatives $\partial_i\partial_j F$ of F. Whereas $\partial_i F(u)$ lies in the tangent space $T_{F(u)}S$ for all $u \in U$, this need not be the case for $\partial_i\partial_j F(u)$. To measure this, we introduce the real-valued functions

$$h_{ij} = \langle \partial_i\partial_j F, \tilde{N}\rangle$$

on U, where $\tilde{N} = N \circ F$. Since $\partial_1\partial_2 F = \partial_2\partial_1 F$ we have

$$h_{12} = h_{21}.$$

Proposition 2.1 *If $u \in U$ and $p = F(u)$ then*

$$h_{ij}(u) = \langle W_p(\partial_i F(u)), \partial_j F(u) \rangle.$$

Proof. Since $\partial_j F(u)$ lies in the tangent space T_pS whereas $\tilde{N}(u)$ is perpendicular to it, we have $\langle \partial_j F, \tilde{N} \rangle = 0$. Differentiating this equality we get

$$0 = \partial_i \langle \partial_j F, \tilde{N} \rangle = \langle \partial_i \partial_j F, \tilde{N} \rangle + \langle \partial_j F, \partial_i \tilde{N} \rangle,$$

hence

$$h_{ij} = -\langle \partial_j F, \partial_i \tilde{N} \rangle.$$

The chain rule gives

$$(\partial_i \tilde{N})(u) = \partial_i (N \circ F)(u) = dN_p(\partial_i F(u)) = -W_p(\partial_i F(u)),$$

from which the proposition follows. □

Corollary 2.1 *The Weingarten map $W_p : T_pS \to T_pS$ is self-adjoint, i.e. for all $v, w \in T_pS$ one has*

$$\langle W_p(v), w \rangle = \langle v, W_p(w) \rangle.$$

Proof. Let $p = F(u)$. The corollary follows because $h_{12} = h_{21}$ and $(\partial_1 F(u), \partial F_2(u))$ is a basis for T_pS. □

As an application of this, let λ_1, λ_2 be the eigenvalues of W_p. Then

$$K(p) = \det(W_p) = \lambda_1 \lambda_2.$$

The components of the first and second fundamental forms make up two symmetric 2×2 matrices $G = (g_{ij})$ and $H = (h_{ij})$. We now express the Gauss curvature K of S in terms of the determinants of these matrices. Let

$$\tilde{K} = K \circ F.$$

Theorem 2.1 $\tilde{K} = \dfrac{\det(H)}{\det(G)}$.

Proof. Let $u \in U$, $p = F(u)$, and $e_i = \partial_i F(u)$. Then (e_1, e_2) is a basis for T_pS, and

$$g_{ij}(u) = \langle e_i, e_j \rangle.$$

Let $A = (a_{ij})$ be the matrix of the Weingarten map W_p with respect to this basis, so that

$$W_p e_j = \sum_i a_{ij} e_i.$$

Then

$$h_{ij} = \langle W_p e_i, e_j \rangle = \langle \sum_k a_{ki} e_k, e_j \rangle = \sum_k g_{jk} a_{ki}.$$

We recognize the last sum as the (ji) entry of the matrix product GA. This means that the transpose of H is

$$H^T = GA,$$

hence

$$\det(H) = \det(H^T) = \det(G)\det(A).$$

Recalling that $K(p) = \det(A)$ and $\det(G) > 0$, this proves the theorem. □

Proposition 2.2 *Let $S \subset \mathbb{R}^3$ be a regular surface. Suppose $p \in S$ and $r > 0$ is a constant such that*

- $\|x\| \leq r$ *for all* $x \in S$,
- $\|p\| = r$.

Then

$$K(p) \geq \frac{1}{r^2}.$$

Proof. Let N be a smooth unit normal field defined in a neighbourhood W of p in S. Let $v \in T_pS$ and choose a smooth curve $\gamma : (-\epsilon, \epsilon) \to W$ such that

$$\gamma(0) = p, \quad \gamma'(0) = v.$$

We consider the function

$$f(t) = \frac{1}{2}\|\gamma(t)\|^2.$$

The first two derivatives are

$$f'(t) = \langle \gamma'(t), \gamma(t) \rangle,$$
$$f''(t) = \langle \gamma''(t), \gamma(t) \rangle + \|\gamma'(t)\|^2.$$

Because f has a maximum at $t = 0$, we have

(i) $0 = f'(0) = \langle v, p \rangle$,

(ii) $0 \geq f''(0) = \langle \gamma''(0), p \rangle + \|v\|^2$.

Since (i) holds for all $v \in T_pS$, we have $p \perp T_pS$, so we may assume that $N(p) = -p/r$. Now observe that

$$0 = \langle \gamma'(t), N(\gamma(t)) \rangle$$

for all t, so

$$0 = \frac{d}{dt}\langle \gamma'(t), N(\gamma(t)) \rangle = \langle \gamma''(t), N(\gamma(t)) \rangle + \langle \gamma'(t), dN_{\gamma(t)}(\gamma'(t)) \rangle.$$

For $t = 0$ we get

$$\langle v, W_p(v) \rangle = \langle \gamma''(0), N(p) \rangle = -\frac{1}{r}\langle \gamma''(0), p \rangle \geq \frac{1}{r}\|v\|^2,$$

where the inequality follows from (ii) above. If v is in fact an eigenvalue of W_p, say $W_p(v) = \lambda v$, then

$$\lambda \|v\|^2 = \langle v, W_p(v) \geq \frac{1}{r}\|v\|^2,$$

so $\lambda \geq 1/r$. Now let λ_1, λ_2 be the eigenvalues of W_p. Then

$$K(p) = \lambda_1 \lambda_2 \geq \frac{1}{r^2}. \qquad \square$$

Corollary 2.2 *If $S \subset \mathbb{R}^3$ is a compact, non-empty surface then there is a point $p \in S$ such that $K(p) > 0$.*

Proof. Let p be a point on S where the function

$$S \to \mathbb{R}, \quad x \mapsto \|x\|^2$$

has a maximum, and let $r = \|p\|$. Since S is a surface, it cannot consist of the origin alone, hence $r > 0$. Therefore,

$$K(p) \geq \frac{1}{r^2} > 0. \qquad \square$$

The following theorem describes a surface locally as the graph of a function.

Theorem 2.2 *Let $S \subset \mathbb{R}^3$ be a regular surface, $p \in S$, and ξ_1, ξ_2, ξ_3 an orthonormal basis for $\mathbb{R}^3$ such that $\xi_1, \xi_2 \in T_pS$.*

(i) *There exists a local parametrization of S around p of the form*

$$F(u_1, u_2) = p + u^1\xi_1 + u^2\xi_2 + f(u^1, u^2)\xi_3,$$

where $f : U \to \mathbb{R}$ is a smooth function satisfying

$$f(0,0) = 0; \quad \partial_i f(0,0) = 0 \text{ for } i = 1, 2.$$

(ii) *If F is any local parametrization as in (i) then the Gauss curvature of S at p agrees with the determinant of the Hessian matrix of f at the origin, i.e.*

$$K(p) = \det(Hess_{(0,0)} f).$$

Proof. (i) Let the maps $\pi, \alpha : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$\pi(\sum_{i=1}^{3} a^i \xi_i) := (a^1, a^2), \quad \alpha(x) := \pi(x - p)$$

for $a^i \in \mathbb{R}$ and $x \in \mathbb{R}^3$. Let

$$\phi := \alpha|_S : S \to \mathbb{R}^2$$

be the restriction of α to S. At any point $x \in S$ the differential of ϕ is the restriction of π, i.e.

$$d_x\phi(v) = \pi(v)$$

for $v \in T_xS$. Therefore, $d_p\phi$ maps the basis ξ_1, ξ_2 for T_pS to the basis $(1,0), (0,1)$ for $\mathbb{R}^2$, so $d_p\phi : T_pS \to \mathbb{R}^2$ is an isomorphism. By the inverse function theorem, ϕ maps some neighbourhood W of p in S to a neighbourhood U of $(0,0)$ in $\mathbb{R}^2$. Let

$$F := \phi^{-1} : U \to W.$$

Because $\alpha \circ F = \mathrm{Id}_U$, there is a smooth function $f : U \to \mathbb{R}$ such that

$$F(u^1, u^2) = p + u^1\xi_1 + u^2\xi_2 + f(u^1, u^2)\xi_3.$$

Since $F(0,0) = p$ we have $f(0,0) = 0$. The partial derivatives of F are

$$\partial_i F = \xi_i + \partial_i f \cdot \xi_3.$$

Because the vectors ξ_1, ξ_2 and $\partial_i F(0,0)$ lie in the tangent space T_pS whereas ξ_3 does not, we must have $\partial_i f(0,0) = 0$.

(ii) Let $G = (g_{ij})$ be the matrix of the first fundamental form. Since $\partial_i F(0,0) = \xi_i$ we see that G is the identity matrix. Choose a smooth normal field N defined in some neighbourhood of p in S such that $N(p) = \xi_3$, and let $H = (h_{ij})$ be the matrix of the second fundamental form relative to N. The second order partial derivatives of F are

$$\partial_i \partial_j F = \partial_i \partial_j f \cdot \xi_3,$$

hence

$$h_{ij}(0,0) = \langle \partial_i \partial_j F(0,0), N(p) \rangle = \partial_i \partial_j f(0,0).$$

Thus, $H(0,0)$ is the Hessian matrix of f at the origin, so

$$K(p) = \frac{\det(H(0,0))}{\det(G(0,0))} = \det(\text{Hess}_{(0,0)} f). \qquad \square$$

If E is any affine plane in $\mathbb{R}^3$ then $\mathbb{R}^3 - E$ has two connected components. A subset $A \subset \mathbb{R}^3$ is said to lie **completely on one side of** E if A is contained in one of the connected components of $\mathbb{R}^3 - E$. From the last theorem we obtain the following corollary.

Corollary 2.3 **(i)** *If $K(p) > 0$ then p has a neighbourhood W in S such that $W - \{p\}$ lies completely on one side of the affine tangent plane $p + T_pS$.*

(ii) *If $K(p) < 0$ then any neighbourhood of p in S contains points from both sides of $p + T_pS$.* $\square$

3 Vector fields

For any vector field X on S and smooth function $h : S \to \mathbb{R}^n$, the **directional derivative**

$$\partial_X h : S \to \mathbb{R}^n$$

is defined by

$$(\partial_X h)(p) := (d_p h)(X_p).$$

Proposition 3.1 *If X is a smooth vector field on the surface S and $h : S \to \mathbb{R}^n$ is smooth then the directional derivative $\partial_X h$ is also smooth.*

Proof. Given $p \in S$, we can find a neighbourhood $V \subset \mathbb{R}^3$ of p and smooth functions

$$\tilde{X} : V \to \mathbb{R}^3, \quad \tilde{h} : V \to \mathbb{R}^n$$

such that on $S \cap V$ we have $\tilde{X} = X$ and $\tilde{h} = h$. Let

$$\tilde{X} = (\tilde{X}^1, \tilde{X}^2, \tilde{X}^3)$$

be the components of $\tilde{X}$. For any point $q \in S \cap V$ we have

$$\partial_X h(q) = d_q h(X_q) = d_q \tilde{h}(\tilde{X}_q) = \sum_i \tilde{X}^i(q) \cdot \partial_i \tilde{h}(q).$$

Since the functions $\tilde{X}^i$ and $\partial_i \tilde{h}$ are smooth, we conclude that $\partial_X h$ is smooth on $S \cap V$. $\square$

Lemma 3.1 *Let S be a regular surface, X a vector field on S. For any smooth functions $f : S \to \mathbb{R}$, and $g, h : S \to \mathbb{R}^n$ the following hold.*

(i) $\partial_X(g + h) = \partial_X g + \partial_X h$.

(ii) $\partial_X(fh) = (\partial_X f)h + f\partial_X h$.

(iii) $\partial_{fX} h = f\partial_X h$.

Proof. Parts (i) and (ii) are left as exercises for the reader. Part (iii) follows from the linearity of the differential $d_p h$ at any point $p \in S$:

$$(\partial_{fX} h)(p) = d_p h(f(p)X(p)) = f(p) \cdot d_p h(X(p)) = (f\partial_X h)(p). \quad \square$$

Lemma 3.2 *For any smooth map $h : S \to \mathbb{R}^n$ and local parametrization (U, F, V) with coordinate vector fields X_1, X_2 the following holds for any i, j.*

(i) $(\partial_{X_i} h) \circ F = \partial_i(h \circ F)$.

(ii) $(\partial_{X_i}\partial_{X_j} h) \circ F = \partial_i\partial_j(h \circ F)$.

(iii) $(\partial_{X_i} X_j) \circ F = \partial_i\partial_j F$.

Proof. (i) For $u \in U$ and $p = F(u)$ we have

$$(\partial_{X_i} h)(p) = d_p h(X_i(p)) = d_p h(\partial_i F(u)) = \partial_i(h \circ F)(u),$$

where the last equality follows from the chain rule.

(ii) Applying (i) twice we get

$$(\partial_{X_i}\partial_{X_j} h) \circ F = \partial_i((\partial_{X_j} h) \circ F) = \partial_i\partial_j(h \circ F).$$

(iii) Take $h = X_j$ in (i). $\square$

Corollary 3.1 $\partial_{X_i} X_j = \partial_{X_j} X_i$.

Proof. This follows from part (iii) of the lemma because $\partial_i \partial_j F = \partial_j \partial_i F$. □

4 Lie brackets

Given smooth vector fields X, Y on a regular surface $S \subset \mathbb{R}^3$, the directional derivative $\partial_X Y$ will in general not be a vector field on S. However, the **Lie bracket**

$$[X, Y] := \partial_X Y - \partial_Y X \tag{1}$$

turns out to be a vector field. This is a consequence of the following proposition, which tells us how to compute the Lie bracket in local coordinates.

Proposition 4.1 *Let X, Y be smooth vector fields on a regular surface S. If X_1, X_2 are coordinate vector fields on an open subset W of S and*

$$X|_W = \sum_i a^i X_i, \quad Y|_W = \sum_i b^i X_i, \tag{2}$$

for (smooth) real-valued functions a^i, b^j on W then

$$[X, Y]|_W = \sum_{ij} (a^i \partial_{X_i} b^j - b^i \partial_{X_i} a^j) X_j.$$

Proof. We calculate

$$(\partial_X Y)|_W = \sum_{ij} a^i \partial_{X_i} (b^j X_j) = \sum_{ij} ((a^i \partial_{X_i} b^j) X_j + a^i b^j \partial_{X_i} X_j).$$

Applying Corollory 3.1 to $\partial_X Y - \partial_Y X$, the terms involving directional derivatives of the coordinate vector fields cancel out, and we obtain the formula in the lemma. □

Example By Corollory 3.1, one has

$$[X_i, X_j] = 0$$

whenever X_1, X_2 are coordinate vector fields on an open set in S.

Proposition 4.2 *For any smooth vector fields X, Y on a regular surface S and smooth function $f : S \to \mathbb{R}$ one has*

$$\partial_{[X,Y]} f = \partial_X \partial_Y f - \partial_Y \partial_X f. \tag{3}$$

Proof. In a neighbourhood of any point in S we can express X and Y in terms of coordinate vector fields as in (2). In that neighbourhood we then have

$$\partial_X \partial_Y f = \sum_{ij} a^i \partial_{X_i}(b^j \partial_{X_j} f) = \sum_{ij} (a^i \partial_{X_i} b^j \cdot \partial_{X_j} f + a^i b^j \partial_{X_i} \partial_{X_j} f).$$

By Lemma 3.2 (ii) we have $\partial_{X_i}\partial_{X_j} f = \partial_{X_j}\partial_{X_i} f$. Applying this to $\partial_X \partial_Y f - \partial_Y \partial_X f$, the terms involving second order directional derivatives cancel out. Comparing the resulting formula with the expression in Proposition 4.1 we obtain (3). $\square$

Proposition 4.3 *Let X, Y be smooth vector fields on a regular surface S and $f : S \to \mathbb{R}$ a smooth function. Prove the following.*

(i) $[fX, Y] = f[X, Y] - (\partial_Y f)X$.

(ii) $[X, fY] = f[X, Y] + (\partial_X f)Y$.

Proof. This follows easily from Lemma 3.1. $\square$

5 The covariant derivative

Let $S \subset \mathbb{R}^3$ be a regular surface. For any $p \in S$ let

$$\Pi_p : \mathbb{R}^3 \to T_pS$$

be the orthogonal projection. Given a function $f : S \to \mathbb{R}^3$, the **tangential part** of f is the vector field $f^{\tan}$ on S defined by

$$f^{\tan}(p) := \Pi_p(f(p)).$$

Proposition 5.1 *If $f : S \to \mathbb{R}^3$ is smooth then the tangential part f^{tan} is also smooth.*

Proof. Given $p \in S$, we can find a smooth unit normal field N defined in a neighbourhood W of p in S. Then on W one has

$$f^{\tan} = f - \langle f, N \rangle N,$$

proving that $f^{\tan}$ is smooth. $\square$

If X, Y are smooth vector fields on S then the **covariant derivative** $\nabla_X Y$ is the smooth vector field on S defined by

$$\nabla_X Y := (\partial_X Y)^{\tan}.$$

One can also define the covariant derivative at a point: If $p \in S$ and $v \in T_pS$ then we define

$$\nabla_{p,v} Y := \Pi_p(d_p Y(v)).$$

If $v = X_p$ we therefore have $(\nabla_X Y)(p) = \nabla_{p,v} Y$.

Proposition 5.2 *For any smooth vector fields X, Y, Z on S and smooth function $f : S \to \mathbb{R}$ one has*

(i) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$

(ii) $\nabla_{fX} Z = f \nabla_X Z$

(iii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

(iv) $\nabla_X (fY) = (\partial_X f) \cdot Y + f \nabla_X Y$

(v) $\nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$

Proposition 5.3 *For any smooth vector fields X, Y, Z on S one has*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Proof. Take horizontal parts on both sides in Definition 1. □

Proposition 5.4 *If X_1, X_2 are coordinate vector fields on an open set in S then*

$$\nabla_{X_i} X_j = \nabla_{X_j} X_i.$$

Proof. This follows from Corollory 3.1 by taking horizontal parts. □

Let X_1, X_2 be coordinate vector fields on an open subset $W \subset S$. Recall that $X_1(p), X_2(p)$ is a basis for the tangent space T_pS for every $p \in W$. Any vector field X on W can therefore be expressed uniquely on the form

$$X = \sum_i a^i X_i$$

for some functions $a^i : W \to \mathbb{R}$. In view of Proposition 5.2, the covariant derivative on W is therefore complete determined by the collection of vector fields $\nabla_{X_i} X_j$. On the other hand,

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

for some smooth functions $\Gamma_{ij}^k : W \to \mathbb{R}$ called **Christoffel symbols**. Note that

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

by Proposition 5.4.

Proposition 5.5 *Let $S \subset \mathbb{R}^3$ be a regular surface with local parametrization (U, F, V) and corresponding coordinate vector fields X_i. Let N be a unit normal field on $S \cap V$. Then*

$$\partial_i \partial_j F = \sum_k \tilde{\Gamma}_{ij}^k \partial_k F + h_{ij} \tilde{N},$$

where

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k \circ F, \quad \tilde{N} = N \circ F$$

and (h_{ij}) are the components of the second fundamental form.

Proof. Let $u \in U$ and $p = F(u) \in S$. Expressing $\partial_{X_i} X_j$ in terms of its tangential and normal parts we get

$$\begin{aligned}
\partial_i \partial_j F(u) &= (\partial_{X_i} X_j)(p) \\
&= (\nabla_{X_i} X_j)(p) + \langle \partial_i \partial_j F(u), N(p) \rangle N(p) \\
&= \sum_k \Gamma_{ij}^k(p) X_k(p) + h_{ij}(u) N(p) \\
&= \sum_k \tilde{\Gamma}_{ij}^k(u) \partial_k F(u) + h_{ij}(u) \tilde{N}(u). \qquad \square
\end{aligned}$$

For the purposes of this section we define the components of the first fundamental form by

$$g_{ij} = \langle X_i, X_j \rangle.$$

Let (g^{ij}) be the inverse matrix of the 2×2 matrix (g_{ij}), so that

$$\sum_j g_{ij} g^{jk} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{else.} \end{cases}$$

Proposition 5.6 $\Gamma_{ij}^k = \frac{1}{2}\sum_\ell g^{kl}(\partial_{X_i} g_{jl} + \partial_{X_j} g_{il} - \partial_{X_\ell} g_{ij})$.

Proof. We calculate

$$\begin{aligned}
\partial_{X_i} g_{jk} &= \partial_{X_i}\langle X_j, X_k\rangle \\
&= \langle \nabla_i X_j, X_k\rangle + \langle X_j, \nabla_i X_k\rangle \\
&= \langle \sum_m \Gamma_{ij}^m X_m, X_k\rangle + \langle X_j, \sum_m \Gamma_{ik}^m X_m\rangle \\
&= \sum_m \left(\Gamma_{ij}^m\, g_{mk} + \Gamma_{ik}^m\, g_{jm}\right).
\end{aligned}$$

We now make cyclic permutations of the indices i, j, k to obtain three equations:

$$\begin{aligned}
\partial_{X_i} g_{jk} &= \sum_m \left(\Gamma_{ij}^m\, g_{mk} + \Gamma_{ik}^m\, g_{jm}\right), \\
\partial_{X_j} g_{ki} &= \sum_m \left(\Gamma_{jk}^m\, g_{mi} + \Gamma_{ji}^m\, g_{km}\right), \\
\partial_{X_k} g_{ij} &= \sum_m \left(\Gamma_{ki}^m\, g_{mj} + \Gamma_{kj}^m\, g_{im}\right).
\end{aligned}$$

Adding the first two equations and subtracting the last one we see that four terms cancel and we are left with

$$\partial_{X_i} g_{jk} + \partial_{X_j} g_{ki} - \partial_{X_k} g_{ij} = 2\sum_m \Gamma_{ij}^m\, g_{mk},$$

which yields

$$\Gamma_{ij}^k = \sum_{\ell m} \Gamma_{ij}^m g^{k\ell} g_{\ell m} = \frac{1}{2}\sum_\ell g^{kl}(\partial_{X_i} g_{jl} + \partial_{X_j} g_{il} - \partial_{X_\ell} g_{ij}). \qquad \square$$

6 Some algebra

Let $E_1, \ldots, E_k, F$ be modules over a ring R. A map

$$T : E_1 \times \cdots \times E_k \to F$$

is called R–**multilinear** (or **multilinear over** R) if it is linear in each variable separately, i.e. if for any $a_i \in E_i$, $i = 1, \ldots, k$ and index j the map

$$E_j \to F, \quad b \mapsto T(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_k)$$

is R–linear.

For any regular surface S, the collection $C^\infty(S)$ of all smooth functions $S \to \mathbb{R}$ is a commutative ring where addition and multiplication are defined pointwise: If $f, g \in C^\infty(S)$ and $p \in S$ then

$$(f+g)(p) = f(p) + g(p), \quad (fg)(p) = f(p)g(p).$$

An example of a module over $C^\infty(S)$ is the collection $\mathfrak{X}(S)$ of all smooth vector fields on S, where addition of vector fields as well as multiplication of a vector field with a function are defined pointwise.

7 The Riemannian curvature tensor

As motivation, we first consider the case when S is an affine plane in $\mathbb{R}^3$. Then $\nabla_X Y = \partial_X Y$ for any smooth vector fields X, Y on S. If Z is a third smooth vector field on S then by applying Proposition 4.2 to each component of Z we get

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \partial_X \partial_Y Z - \partial_Y \partial_X Z = \partial_{[X,Y]} Z = \nabla_{[X,Y]} Z.$$

For an arbitrary regular surface S in $\mathbb{R}^3$, the Riemannian curvature tensor associates to every triple X, Y, Z of smooth vector fields on S the smooth vector field

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Thus, if S is an affine plane then $R = 0$. We are going to show that the Riemannian curvature tensor is preserved by local isometries, hence it provides a measure of how much a given surface deviates from being locally isometric to a plane. We will also express the Gauss curvature K in terms of R, proving that Gauss curvature is also preserved by local isometries. (This is the famous *Theorema Egregium* of Gauss.)

Proposition 7.1 *The map*

$$\mathfrak{X}(S) \times \mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}(S), \quad (X, Y, Z) \mapsto R(X,Y)Z$$

is multilinear over $C^\infty(S)$.

Proof. This is a straightforward application of Propositions 4.3 and 5.2. Additivity in each variable is obvious. Now let $f \in C^\infty(S)$. Then

$$\begin{aligned} R(fX,Y)Z &= f\nabla_X \nabla_Y Z - \nabla_Y (f\nabla_X Z) - \nabla_{f[X,Y] - \partial_Y f \cdot X}(Z) \\ &= f\nabla_X \nabla_Y Z - \partial_Y f \cdot \nabla_X Z - f\nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z + \partial_Y f \cdot \nabla_X Z \\ &= fR(X,Y)Z. \end{aligned}$$

Furthermore,

$$R(X, fY)Z = -R(fY, X)Z = -fR(Y, X)Z = fR(X, Y)Z.$$

The proof that $R(X, Y)(fZ) = fR(X, Y)Z$ is left as an exercise. □

Proposition 7.2 *For all smooth vector fields X, Y, Z, W on S one has*

$$\langle R(X, Y)Z, W\rangle = -\langle Z, R(X, Y)W\rangle.$$

Proof. We calculate

$$\begin{aligned}\partial_X\partial_Y\langle Z, W\rangle &= \partial_X(\langle\nabla_Y, W\rangle + \langle Z, \nabla_Y, W\rangle) \\ &= \langle\nabla_X\nabla_Y Z, W\rangle + \langle\nabla_Y Z, \nabla_X W\rangle \\ &+ \langle\nabla_X Z, \nabla_Y W\rangle + \langle Z, \nabla_X\nabla_Y W\rangle).\end{aligned}$$

In the final expression, the sum of the second and third terms is symmetric in X and Y. If we make the same calculation with X and Y reversed and subtract the results, the terms involving first-order covariant derivatives therefore cancel out, and we obtain the following.

$$\begin{aligned}\partial_{[X,Y]}\langle Z, W\rangle &= \partial_X\partial_Y\langle Z, W\rangle - \partial_Y\partial_X\langle Z, W\rangle \\ &= \langle\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z, W\rangle + \langle Z, \nabla_X\nabla_Y W - \nabla_Y\nabla_X W\rangle.\end{aligned}$$

Combining this with

$$\partial_{[X,Y]}\langle Z, W\rangle = \langle\nabla_{[X,Y]}Z, W\rangle + \langle Z, \nabla_{[X,Y]}W\rangle$$

we obtain the proposition. □

The previous proposition makes it possible to define the Riemannian curvature R_p at any point p in S, as we now explain. For any finite-dimensional real vector space V equipped with a scalar product let $\mathrm{so}(V)$ denote the space of all skew-symmetric endomorphisms of V, i.e. linear maps $A : V \to V$ such that

$$\langle Ax, y\rangle = -\langle x, Ay\rangle$$

for all $x, y \in V$.

Proposition 7.3 *Let S be a regular surface. For each point $p \in S$ there is a unique skew-symmetric bilinear map*

$$R_p : T_pS \times T_pS \to so(T_p(S))$$

such that for any smooth vector fields X, Y, Z defined in a neighbourhood of p in S one has

$$[R(X,Y)Z]_p = R_p(X_p, Y_p)Z_p.$$

Proof. The map R_p is unique because for every tangent vector $v \in T_pS$ there exists a smooth vector field X defined in some neighbourhood of p in S such that $X_p = v$. To prove existence, let X_1, X_2 be coordinate vector fields in some neighbourhood W of p in S. We consider three smooth vector fields on W given as

$$X = \sum_i a^i X_i, \quad Y = \sum_j b^j X_j, \quad Z = \sum_k c^k X_k,$$

where a^i, b^j, c^k are smooth functions on W. Then

$$R(X,Y)Z = \sum_{ijk} a^i b^j c^k R(X_i, X_j)X_k.$$

We can therefore define R_p in terms of the basis $X_1(p), X_2(p)$ for T_pS by

$$R_p(X_i(p), X_j(p))X_k(p) := [R(X_i, X_j))X_k](p). \qquad \square$$

Given coordinate vector fields X_1, X_2 on an open subset W of a regular surface S, there are smooth, real-valued functions R^ℓ_{ijk} on W such that

$$R(X_i, X_j)X_k = \sum_\ell R^\ell_{ijk} X_\ell.$$

These functions R^ℓ_{ijk} are called the *components* of the curvature tensor.

Proposition 7.4 $R^\ell_{ijk} = \partial_{X_i}\Gamma^\ell_{jk} - \partial_{X_j}\Gamma^\ell_{ik} + \sum_m \Big(\Gamma^m_{jk}\Gamma^\ell_{im} - \Gamma^m_{ik}\Gamma^\ell_{jm}\Big).$

Proof. This is a straight-forward calculation:

$$\begin{aligned}\nabla_{X_i}\nabla_{X_j}X_k &= \nabla_{X_i}\sum_\ell \Gamma^\ell_{jk}X_\ell \\ &= \sum_\ell \Big(\partial_{X_i}\Gamma^\ell_{jk}\cdot X_\ell + \Gamma^\ell_{jk}\nabla_{X_i}X_\ell\Big) \\ &= \sum_\ell \partial_{X_i}\Gamma^\ell_{jk}\cdot X_\ell + \sum_{\ell m}\Gamma^\ell_{jk}\Gamma^m_{i\ell}X_m.\end{aligned}$$

Interchanging ℓ and m in the last sum we obtain

$$\nabla_{X_i}\nabla_{X_j}X_k = \sum_\ell \left(\partial_{X_i}\Gamma^\ell_{jk} + \sum_m \Gamma^m_{jk}\Gamma^\ell_{im} \right) X_\ell.$$

Now recall that $[X_i, X_j] = 0$, hence

$$R(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k.$$

Putting this together, we get the formula of the proposition. □

8 Theorema Egregium

Our next goal is to express the Gauss curvature of a regular surface in terms of the Riemannian curvature tensor. This will lead to a proof of Gauss's *Theorema Egregium* (remarkable theorem), which asserts that the Gauss curvature is preserved by local isometries.

Let $S \subset \mathbb{R}^3$ be a regular surface. The **normal part** of a function $f : S \to \mathbb{R}^3$ is the normal field f^{nor} on S defined by

$$f^{\text{nor}} := f - f^{\tan}.$$

Given smooth vector fields X, Y on S we define the normal field

$$\alpha(X, Y) := (\partial_X Y)^{\text{nor}},$$

so that

$$\partial_X Y = \nabla_X Y + \alpha(X, Y)$$

is the decomposition of $\partial_X Y$ into its tangential and normal parts.

Proposition 8.1 $\alpha(X, Y) = \alpha(Y, X)$.

Proof. By definition of the Lie bracket we have

$$[X, Y] = \partial_X Y - \partial_Y X.$$

Because $[X, Y]$ is a vector field, its normal part is zero, hence

$$0 = [X, Y]^{\text{nor}} = (\partial_X Y)^{\text{nor}} - (\partial_Y X)^{\text{nor}} = \alpha(X, Y) - \alpha(Y, X). \qquad \square$$

Let $\mathfrak{X}^\perp(S)$ be the set of all smooth normal fields on S, which is a module over the ring $C^\infty(S)$ of smooth functions on S.

Proposition 8.2 *The map*

$$\alpha : \mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}(S)^{\perp}$$

is bilinear over $C^{\infty}(S)$.

Proof. Biadditivity of α is obvious. For $f \in C^{\infty}(S)$ we have

$$\alpha(fX, Y) = (\partial_{fX} Y)^{\text{nor}} = (f \partial_X Y)^{\text{nor}} = f\alpha(X, Y).$$

By symmetry of α we also have $\alpha(X, fY) = f\alpha(X, Y)$. $\square$

Proposition 8.3 *Let S be a regular surface. For any point $p \in S$ there is a unique symmetric bilinear map*

$$\alpha_p : T_pS \times T_pS \to (T_pS)^{\perp}$$

such that if X, Y are smooth vector fields defined in some neighbourhood of p in S then

$$[\alpha(X, Y)]_p = \alpha_p(X_p, Y_p).$$

Proof. This is proved in the same way as the corresponding statement for the Riemannian curvature tensor, see Proposition 7.3. $\square$

Proposition 8.4 *If $N : S \to \mathbb{R}^3$ is a smooth unit normal field and $p \in S$ then for all tangent vectors $v, w \in T_pS$ one has*

$$\alpha_p(v, w) = I\!I_p(v, w) \cdot N_p,$$

where $I\!I_p$ is the second fundamental form relative to N.

Proof. Choose smooth vector fields X, Y defined in some neighbourhood of p in S such that $X_p = v$ and $Y_p = w$. Then

$$0 = \partial_X \langle Y, N \rangle = \langle \partial_X Y, N \rangle + \langle Y, \partial_X N \rangle.$$

Evaluating at p we get

$$\begin{aligned} \langle \alpha_p(v, w), N_p \rangle &= \langle \partial_X Y, N \rangle_p = -\langle Y, \partial_X N \rangle_p \\ &= -\langle w, d_pN(v) \rangle = \langle w, W_p(v) \rangle = I\!I_p(v, w), \end{aligned}$$

where W_p is the Weingarten map. $\square$

Theorem 8.1 (Gauss equation) *For all smooth vector fields X, Y, Z, W on S one has*

$$-\langle R(X,Y)Z, W\rangle = \langle \alpha(X,Z), \alpha(Y,W)\rangle - \langle \alpha(X,W), \alpha(Y,Z)\rangle.$$

Proof. We begin by calculating

$$\begin{aligned}\langle \partial_X \partial_Y, Z, W\rangle &= \langle \partial_X(\nabla_Y Z + \alpha(Y,Z)), W\rangle \\ &= \langle \nabla_X \nabla_Y, W\rangle + \langle \partial_X \alpha(Y,Z), W\rangle.\end{aligned}$$

On the other hand,

$$0 = \partial_X \langle \alpha(Y,Z), W\rangle = \langle \partial_X \alpha(Y,Z), W\rangle + \langle \alpha(Y,Z), \partial_X W\rangle,$$

hence

$$\langle \partial_X \alpha(Y,Z), W\rangle = -\langle \alpha(Y,Z), \alpha(X,W)\rangle.$$

Altogether, we obtain

$$\langle \partial_X \partial_Y Z, W\rangle = \langle \nabla_X \nabla_Y, W\rangle - \langle \alpha(Y,Z), \alpha(X,W)\rangle.$$

Finally,

$$\begin{aligned}\langle \nabla_{[X,Y]} Z, W\rangle &= \langle \partial_X \partial_Y Z - \partial_Y \partial_X Z, W\rangle \\ &= \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z, W\rangle \\ &\quad - \langle \alpha(Y,Z), \alpha(X,W)\rangle + \langle \alpha(X,Z), \alpha(Y,W)\rangle,\end{aligned}$$

from which the theorem follows. □

Theorem 8.2 *Let $S \subset \mathbb{R}^3$ be a regular surface with Gauss curvature K and Riemannian curvature tensor R. Then for any $p \in S$ and orthonormal basis v_1, v_2 for T_pS one has*

$$K(p) = -\langle R_p(v_1, v_2)v_1, v_2\rangle.$$

Proof. Let (a_{ij}) be the matrix of the Weingarten map $W_p : T_pS \to T_pS$ with respect to the basis v_1, v_2. Then

$$a_{ij} = \langle v_i, W_p(v_j)\rangle = II(v_i, v_j).$$

By Proposition 8.4 we have

$$\alpha_p(v_i, v_j) = a_{ij} N_p.$$

Since $\langle N_p, N_p\rangle = 1$, the Gauss equation yields

$$-\langle R_p(v_1, v_2)v_1, v_2\rangle = a_{11}a_{22} - a_{12}^2 = \det W_p = K(p). \qquad \square$$

Theorem 8.3 (Theorema Egregium) *If $\phi : S \to \bar{S}$ is an isometry between regular surfaces with Gauss curvatures $K, \bar{K}$, respectively, then*

$$K = \bar{K} \circ \phi.$$

Proof. Let $p \in S$ and $\bar{p} = \phi(p)$. We must show that $K(p) = \bar{K}(\bar{p})$. Let $F : U \to \mathbb{R}^3$ be a local parametrization of S around p. Then $\bar{F} := \phi \circ F : U \to \mathbb{R}^3$ is a local parametrization of $\bar{S}$. Let g_{ij}, Γ_{ij}^k, R_{ijk}^ℓ, X_i be the components of the first fundemental form, Christoffel symbols, components of the curvature tensor, and coordinate vector fields defined by F. Let $\bar{g}_{ij}$, $\bar{\Gamma}_{ij}^k$, $\bar{R}_{ijk}^\ell$, and $\bar{X}_i$ be the corresponding quantities defined by $\bar{F}$.

Suppose $p = F(u)$, $u \in U$. The chain rule yields

$$AX_i(p) = d_p\phi(\partial_i F(u)) = \partial_i(\phi \circ F)(u) = \partial_i \bar{F}(u) = \bar{X}_i(\bar{p}).$$

Because the differential $A := d_p\phi$ is an isometry,

$$g_{ij}(p) = \langle X_i(p), X_j(p) \rangle = \langle AX_i(p), AX_j(p) \rangle = \langle \bar{X}_i(\bar{p}), \bar{X}_j(\bar{p}) \rangle = \bar{g}_{ij}(\bar{p}).$$

Proposition 5.6 then implies that $\Gamma_{ij}^k(p) = \bar{\Gamma}_{ij}^k(\bar{p})$, and Proposition 7.4 yields $R_{ijk}^\ell(p) = \bar{R}_{ijk}^\ell(\bar{p})$. Given tangent vectors $v_1, v_2, v_3 \in T_pS$, the equation

$$A(R_p(v_1, v_2)v_3) = \bar{R}_p(Av_1, Av_2)Av_3$$

therefore holds whenever each v_i is one of the basis vectors $X_j(p)$. By multilinearity of R_p, the same equation holds for all v_i. If v_1, v_2 is an orthonormal basis for T_pS, then Av_1, Av_2 is an orthonormal basis for $T_{\bar{p}}\bar{S}$, and by Theorem 8.2 we have

$$\begin{aligned} K(p) &= -\langle R_p(v_1, v_2)v_1, v_2 \rangle = -\langle A(R_p(v_1, v_2)v_1), Av_2 \rangle \\ &= -\langle R_p(Av_1, Av_2)Av_1, Av_2 \rangle = \bar{K}(\bar{p}). \qquad \square \end{aligned}$$

9 Submanifols of R^n

For non-negative integers k, n, a subset $M \subset \mathbb{R}^n$ is called a k–**dimensional submanifold** if for every point $p \in S$ there is an open set $U \subset \mathbb{R}^k$ and a smooth map $F : U \to \mathbb{R}^n$ such that

(i) F maps U homeomorphically onto a neighbourhood of p in M, and

(ii) For any $u \in U$ the derivative $d_uF : \mathbb{R}^k \to \mathbb{R}^n$ is injective.

Such a map F is called a **local parametrization** of M, and the inverse map $F(U) \to U$ is called a **chart** on M. By a **manifold** we will mean a submanifold of some Euclidean space $\mathbb{R}^n$. A 2–dimensional submanifold of $\mathbb{R}^3$ is called a **regular surface**.

The notion of a smooth map between manifolds is defined just as for maps between regular surfaces. Tangent spaces, differentials of smooth maps, vector fields, and Lie brackets are also defined as before.

10 Differential forms

For $\ell \geq 1$, a **differential form** on M of degree ℓ is a rule ϕ that assigns to every point $p \in M$ a multilinear alternating map

$$\phi_p : \underbrace{T_pM \times \cdots T_pM}_{\ell \text{ times}} \to \mathbb{R}.$$

By alternating we mean that for every permutation σ of the set $\{1, \ldots, \ell\}$ and all tangent vectors $v_1, \ldots, v_\ell \in T_pM$ one has

$$\phi_p(v_{\sigma(1)}, \ldots, v_{\sigma(\ell)}) = \text{sgn}(\sigma)\phi_p(v_1, \ldots, v_\ell),$$

where $\text{sgn}(\sigma) = \pm 1$ is the sign of the permutation. By a differential form on M of degree 0 we simply mean a real-valued function on M. Differential forms of degree ℓ are often called **ℓ–forms**. An ℓ–form ϕ on M is **smooth** if for all smooth vector fields $X_1, \ldots, X_\ell$ on M the function

$$\phi(X_1, \ldots, X_\ell) : M \to \mathbb{R}, \quad p \mapsto \phi_p((X_1)_p, \ldots, (X_\ell)_p)$$

is smooth. The set $\Omega^\ell(M)$ of all smooth ℓ–forms on M is a module over the ring $C^\infty(M)$ of smooth functions on M.

Note that a 1–form α assigns to every $p \in M$ a linear map $\alpha_p : T_pM \to \mathbb{R}$, whereas a 2–form β assigns to every p a bilinear skew-symmetric map

$$\beta_p : T_pM \times T_pM \to \mathbb{R}.$$

For any real vector space V let $A_2(V)$ denote the real vector space of all bilinear skew-symmetric maps $V \times V \to \mathbb{R}$.

Lemma 10.1 *If V has dimension 2 then $A_2(V)$ has dimension 1.*

Proof. Let e_1, e_2 be a basis for V, and $f \in A_2(V)$. Given elements $v, w \in V$ represented as

$$v = v^1 e_1 + v^2 e_2, \quad w = w^1 e_1 + w^2 e_2,$$

where $v^i, w^j \in \mathbb{R}$, we have

$$f(v, w) = \sum_{ij} v^i w^j f(e_i, e_j) = (v^1 w^2 - v^2 w^1) f(e_1, e_2).$$

This shows that the map

$$A_2(V) \to \mathbb{R}, \quad f \mapsto f(e_1, e_2)$$

is injective. It is also surjective, because for any $t \in \mathbb{R}$ the map

$$V \times V \to \mathbb{R}, \quad (v, w) \mapsto t(v^1 w^2 - v^2 w^1)$$

belongs to $A_2(V)$. □

The **wedge product**

$$\Omega^\ell(M) \times \Omega^m(M) \to \Omega^{\ell+m}(M), \quad (\phi, \psi) \mapsto \phi \wedge \psi$$

is a $C^\infty(M)$–bilinear map defined for all non-negative integers ℓ, m, see [5, 3]. We define it here for $\ell = m = 1$. Given $\phi, \psi \in \Omega^1(M)$ we define $\phi \wedge \psi \in \Omega^2(M)$ by

$$(\phi \wedge \psi)_p(v, w) := \phi_p(v)\psi_p(w) - \phi_p(w)\psi_p(v)$$

for $p \in M$ and $v, w \in T_pM$. For vector fields X, Y on M one then has

$$(\phi \wedge \psi)(X, Y) = \phi(X)\psi(Y) - \phi(Y)\psi(X).$$

11 The exterior derivative

The **exterior derivative**

$$d : \Omega^\ell(M) \to \Omega^{\ell+1}(M)$$

is a real-linear map defined for all $\ell \geq 0$, see [5, 3]. We define it here for $\ell = 0, 1$.

Given $f \in \Omega^0(M) = C^\infty(M)$, the 1–form df on M is defined by

$$(df)_p(v) := d_p f(v),$$

for $p \in M$, $v \in T_pM$. Here, $d_p f : T_pM \to \mathbb{R}$ is the differential of f at p. For any smooth vector field X on M we then have

$$(df)(X) = \partial_X f.$$

Proposition 11.1 *For any smooth* 1*–form* α *on* M *there is a unique smooth* 2*–form* $d\alpha$ *on* M *such that for all smooth vector fields* X, Y *on* M *one has*

$$d\alpha(X,Y) = \partial_X(\alpha(Y)) - \partial_Y(\alpha(X)) - \alpha([X,Y]). \tag{4}$$

Proof. We claim that right hand side of Equation (4) defines a $C^\infty(M)$–bilinear map

$$B : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M).$$

Given this, we can complete the proof of the proposition by arguing as in the proof of Proposition 7.3.

The map B is obviously biadditive. Now let $f \in C^\infty(M)$. Then

$$\begin{aligned} B(fX,Y) &= \partial_{fX}\alpha(Y) - \partial_Y\alpha(fX) - \alpha([fX,Y]) \\ &= f\partial_X\alpha(Y) - \partial_Y(f\cdot\alpha(X)) - \alpha(f[X,Y] - \partial_Y f\cdot X) \\ &= f\partial_X\alpha(Y) - \partial_Y f\cdot\alpha(X) - f\partial_Y\alpha(X) - f\alpha([X,Y]) + \partial_Y f\cdot\alpha(X) \\ &= f\cdot B(X,Y). \end{aligned}$$

Because B is skew-symmetric, we also have $B(X,fY) = f\cdot B(X,Y)$. □

Proposition 11.2 *For any* $f, g \in \Omega^0(M)$ *one has*

$$d(fdg) = df \wedge dg.$$

Proof. For all smooth vector fields X, Y on M one has

$$\begin{aligned} [d(fdg)](X,Y) &= \partial_X(f\partial_Y g) - \partial_Y(f\partial_X g) - f\partial_{[X,Y]}g \\ &= \partial_X f\cdot\partial_Y x + f\partial_X\partial_Y g - \partial_Y f\cdot\partial_X g - f\partial_Y\partial_X g - f\partial_{[X,Y]}g \\ &= (df \wedge dg)(X,Y), \end{aligned}$$

where in the last equation we used Proposition 4.2, which holds on any manifold. □

Let $x^1, \ldots, x^k$ be standard coordinates on $\mathbb{R}^k$. The ith coordinate x^i is a smooth map $\mathbb{R}^n \to \mathbb{R}$ whose differential $dx^i \in \Omega^1(\mathbb{R}^k)$ is given by

$$(dx^i)_p(v) = v^i$$

for any tangent vector $v = (v^1, \ldots, v^k) \in T_p\mathbb{R}^k = \mathbb{R}^k$. On an open subset $U \subset \mathbb{R}^k$, any smooth 1–form α therefore has the form

$$\alpha = \sum_i f_i dx^i$$

for some $f_i \in C^\infty(U)$, and by Proposition 11.2 we have

$$d\alpha = \sum_i df_i \wedge dx^i.$$

12 Volume forms and orientations

Let $S \subset \mathbb{R}^3$ be an oriented regular surface with smooth unit normal field $N : S \to \mathbb{R}^3$. The **(Riemannian) volume form** on S is the smooth 2–form μ defined by

$$\mu_p(v, w) := \det(v, w, N_p) \tag{5}$$

for $v, w \in T_pS$.

Lemma 12.1 *If μ is the volume form of an oriented surface S then*

$$\mu_p(v, w) = \pm 1$$

for any orthonormal basis (v, w) for T_pS.

Proof. This holds because the 3×3 matrix with columns v, w, N_p is orthogonal and therefore has determinant ± 1. □

Conversely, any smooth 2–form μ on S satisfying the conclusion of the lemma determines an orientation of S through the formula (5).

If S has volume form μ then an ordered basis (v, w) for T_pS is called **positively oriented** if $\mu_p(v, w) > 0$; otherwise it is called **negatively oriented**.

13 Frames

Let $S \subset \mathbb{R}^3$ be a regular surface. A **frame** on an open subset $V \subset S$ is a pair (E_1, E_2) of vector fields on V such that $(E_1(p), E_2(p))$ is a basis for T_pS for every $p \in V$. The frame is **smooth** if each E_i is smooth. By a **local frame** on S we mean a frame on some open subset of S.

Example If $F : U \to S$ is a local parametrization then the associated coordinate vector fields X_1, X_2 form a smooth frame on $F(U)$.

A frame (E_1, E_2) on $V \subset S$ is **orthonormal** if $(E_1(p), E_2(p))$ is an orthonormal basis for T_pS for every $p \in V$. Note that applying the Gram-Schmidt process to an arbitrary frame produces an orthonormal frame. Hence, there is a smooth orthonormal frame on a neighbourhood of any point on S.

If S is oriented then a frame (E_1, E_2) on $V \subset S$ is **positively oriented** if $(E_1(p), E_2(p))$ is a positively oriented basis for T_pS for every $p \in V$; otherwise the frame is **negatively oriented**.

14 Connection forms

Let $S \subset \mathbb{R}^3$ be a regular surface. To any smooth frame (E_1, E_2) on an open subset $V \subset S$ we can associate a 2×2 matrix (ω_i^j) of smooth 1–forms on V called **connection forms**. These are uniquely determined by the fact that

$$\nabla_X E_i = \sum_j \omega_i^j(X) \cdot E_j$$

for any vector field X on V.

Lemma 14.1 *If the frame (E_1, E_2) is orthonormal then the matrix (ω_i^j) is skew-symmetric, i.e.*

$$\omega_i^j = -\omega_j^i$$

for all i, j.

Proof. Because $\langle E_i, E_j \rangle$ is a constant function on V we have

$$0 = \partial_X \langle E_i, E_j \rangle = \langle \nabla_X E_i, E_j \rangle + \langle E_i, \nabla_X E_j \rangle = \omega_i^j(X) + \omega_j^i(X). \qquad \square$$

This means that the matrix (ω_i^j) is completely determined by the element ω_2^1, which we simply denote by ω and refer to as the **connection form** of the frame. We then have

$$\begin{aligned} \nabla_X E_1 &= \omega_1^2(X) E_2 = -\omega(X) E_2, \\ \nabla_X E_2 &= \omega_2^1(X) E_1 = \omega(X) E_1 \end{aligned}$$

for any vector field X on V.

Proposition 14.1 *Let $S \subset \mathbb{R}^3$ be an oriented surface with Gauss curvature K and volume form μ. Let (E_1, E_2) be a positively oriented, orthonormal frame on an open subset $V \subset S$ and ω the corresponding connection form. Then*

$$d\omega = K\mu.$$

Proof. For any smooth vector fields X, Y on V we have

$$\begin{aligned} d\omega(X, Y) &= \partial_X \omega(Y) - \partial_Y \omega(X) - \omega([X, Y]) \\ &= \partial_X \langle \nabla_Y E_2, E_1 \rangle - \partial_Y \langle \nabla_X E_2, E_1 \rangle - \langle \nabla_{[X,Y]} E_2, E_1 \rangle \\ &= \langle \nabla_X \nabla_Y E_2, E_1 \rangle + \langle \nabla_Y E_2, \nabla_X E_1 \rangle \\ &\quad - \langle \nabla_Y \nabla_X E_2, E_1 \rangle - \langle \nabla_X E_2, \nabla_Y E_1 \rangle - \langle \nabla_{[X,Y]} E_2, E_1 \rangle \\ &= \langle R(X, Y) E_2, E_1 \rangle. \end{aligned}$$

By Theorem 8.2 we therefore have

$$K = \langle R(E_1, E_2)E_2, E_1\rangle = d\omega(E_1, E_2).$$

By Lemma 10.1 we can write $d\omega = f\mu$ for some real-valued function f on V. Then $f = d\omega(E_1, E_2) = K$, and the proposition is proved. □

15 Line integrals

Let M be a manifold and $\alpha \in \Omega^1(M)$. For any smooth curve $c : [a, b] \to M$ we define

$$\int_c \alpha := \int_a^b \alpha_{c(t)}(\dot{c}(t))dt.$$

Lemma 15.1 *Let α be a smooth 1–form on M and $c : [a, b] \to M$ a smooth curve. If $\phi : [a', b'] \to [a, b]$ is a smooth function such that $\phi(a') = a$ and $\phi(b') = b$ then*

$$\int_c \alpha = \int_{c\circ\phi} \alpha.$$

Proof. Exercise. □

16 Surface integrals

Let M be a manifold. A curve $c : I \to M$ is called **regular** if c is smooth and $\dot{c}(t) \neq 0$ for all $t \in I$. A continuous, non-constant curve $c : \mathbb{R} \to M$ is called **periodic** if there exists a positive real number λ such that

$$c(t + \lambda) = c(t)$$

for all t. The smallest such λ is then called the **period** of c.

Example The plane curve $c(t) = (\cos t, \sin t)$ has period 2π.

For given $L > 0$, curves $c : \mathbb{R} \to M$ of period L are in one-to-one correspondence with maps $f : S^1 \to M$ through the relation

$$c(t) = f(e^{2\pi it/L}).$$

Moreover, c is smooth if and only if f is smooth. If f is injective, or equivalently if c restricts to an injective map $[0, L) \to M$, then c is called **simple periodic**. In this case, f is a topological embedding. If in addition c is regular then one can show that f is a diffeomorphism onto a submanifold of M, see [5, 3].

Now let S be a regular surface and $c : I \to S$ a regular curve. By a **normal orientation** of c we mean a smooth map $N : I \to S^2$ such that $N(t) \in T_{c(t)}S$ and $N(t) \perp \dot{c}(t)$ for all t. (In particular, N is a vector field on S along c.)

By a **smooth region** in S we mean a compact subset $R \subset S$ which is the closure (in S) of an open subset of S and whose boundary ∂R is the image of a simple periodic, regular curve $c : \mathbb{R} \to S$. In this case, the curve c has a canonical normal orientation N such that $N(t)$ is inward-pointing with respect to R for every t. (One can show that R is a 2–manifold-with-boundary, and a precise definition of inward-pointing is then given in [5].) If in addition S is oriented, we say c is **positively oriented** with respect to R if $(\dot{c}(t), N(t))$ is a positively oriented basis for $T_{c(t)}S$ for every t. If c is positively oriented and has period L then for $\omega \in \Omega^1(S)$ the integral

$$\int_{\partial R} \omega := \int_0^L \omega_{c(t)}(\dot{c}(t))\, dt$$

is easily seen to be independent of the choice of c.

For a regular surface S (oriented or not) we refer to [1] for the definition of the surface integral $\int_S f\, dA$ for integrable functions $f : S \to \mathbb{R}$. If S is oriented with volume form μ then any 2–form ϕ on S can be expressed as $\phi = f\mu$ for a unique function $f : S \to \mathbb{R}$, and we define

$$\int_S \phi := \int_S f\, dA.$$

A definition of $\int_S \phi$ which makes no reference to Riemannian metrics can be found in [5].

Theorem 16.1 (Stokes) *Let S be an oriented regular surface and $R \subset S$ a smooth region. For any $\omega \in \Omega^1(S)$ one then has*

$$\int_{\partial R} \omega = \int_R d\omega.$$

If S is the xy–plane with the standard orientation then $\omega = f\, dx + g\, dy$ for some smooth functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ and

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy.$$

Stokes's theorem now says that

$$\int_{\partial R} (f\,dx + g\,dy) = \int_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy,$$

which is an instance of Green's theorem.

17 Winding numbers

In this section we will state the Hopf *Umlaufsatz*, or rotation index theorem, which will be used in the proof of the Gauss Bonnet theorem.

We will make use of the complex exponential function e^z. Recall that if $z = x + iy$ for real numbers x, y then

$$e^z = e^x(\cos y + i \sin y).$$

Lemma 17.1 *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{C} - \{0\}$ a continuously differentiable function.*

(i) There exists a continuously differentiable function $g : I \to \mathbb{C}$ such that $f(t) = e^{g(t)}$ for all $t \in I$.

(ii) If g_1, g_2 are two functions as in (i) then

$$g_1 - g_2 = 2\pi i k$$

for some constant $k \in \mathbb{Z}$.

Proof. Choose $t_0 \in I$ and a complex number a such that $f(t_0) = e^a$. To prove (ii), suppose $f = e^g$. Then

$$g(t_0) = a + 2\pi i k$$

for some integer k. Moreover,

$$\dot{f} = \dot{g}e^g = \dot{g}f,$$

so $\dot{g} = \dot{f}/f$. Therefore,

$$g(t) = g(t_0) + \int_{t_0}^{t} \dot{g} = a + 2\pi i k + \int_{t_0}^{t} \frac{\dot{f}}{f},$$

proving (ii).

To prove (i), define

$$g(t) := a + \int_{t_0}^{t} \frac{\dot f}{f}.$$

Then $\dot g = \dot f/f$. Writing $h := fe^{-g}$ we have

$$\dot h = \dot f e^{-g} - f\dot g e^{-g} = 0,$$

hence h is constant. Because $h(t_0) = 1$, we have $h \equiv 1$, so $f = e^g$. □

Let $c : \mathbb{R} \to \mathbb{C}$ be a continuously differentiable curve with period L, and z_0 a complex number not in the image of c. The **winding number** $W(c; z_0)$ of c with respect to z_0 is defined as follows. By Lemma 17.1 we can find a continuously differentiable curve $g : \mathbb{R} \to \mathbb{C}$ such that

$$c(t) = z_0 + e^{g(t)}$$

for all t. Then

$$g(t + L) = g(t) + 2\pi ik$$

for some constant integer k, and we define $W(c; z_0) := k$. Part (ii) of the lemma shows that this definition is independent of the choice of g.

Note that if $c(t) = z_0 + r(t)e^{i\theta(t)}$ for real-valued functions r, θ with $r > 0$ then

$$W(c; z_0) = \frac{1}{2\pi}(\theta(L) - \theta(0)).$$

Let $c : \mathbb{R} \to \mathbb{C}$ be a regular, periodic curve. The **rotation index** n_c of c (also called the **tangent winding number**) is the winding number of the derivative $\dot c : \mathbb{R} \to \mathbb{C}$ with respect to the origin, i.e.

$$n_c := W(\dot c; 0).$$

If c is in fact simple periodic then one can show that its image C is a submanifold of $\mathbb{R}^2$ diffeomorphic to S^1. The Jordan curve theorem then asserts that the complement $\mathbb{R}^2 - C$ has exactly two connected components, and C is their common boundary. (A proof of the more general Jordan-Brouwer separation theorem can be found in [2, p. 89].) Moreover, one component (the "inside") is bounded, whereas the other one (the "outside") is unbounded. We say c is **positively oriented** if it is positively oriented with respect to the closure R of the bounded component.

Theorem 17.1 (Hopf) *Any positively oriented, regular, simple periodic curve in the plane has rotation index* 1.

For the proof we refer to [1, 4].

18 Geodesic curvature

Let S be a regular surface and $\gamma : I \to S$ a smooth curve of unit speed and with normal orientation N. Because

$$0 = \frac{d}{dt}\|\dot{\gamma}(t)\|^2 = 2\left\langle \frac{\nabla}{dt}\dot{\gamma}(t), \dot{\gamma}(t) \right\rangle,$$

there is a unique smooth function $\kappa_\gamma : I \to \mathbb{R}$, the **geodesic curvature** of γ, such that

$$\frac{\nabla}{dt}\dot{\gamma}(t) = \kappa_\gamma(t) \cdot N(t)$$

for all t. Clearly, $\kappa_\gamma \equiv 0$ if and only if γ is a geodesic.

The following lemma says that the geodesic curvature is invariant under reparametrization in a certain sense.

Lemma 18.1 *Let $I, J \subset \mathbb{R}$ be intervals. Let $\gamma_1 : I \to S$ be a smooth curve of unit speed and with normal orientation N. Suppose $\gamma_2 = \gamma_1 \circ \phi : J \to S$ is a reparametrization of γ_1 of unit speed, where $\phi : J \to I$ is smooth. Let γ_2 have the normal orientation $N_2(t) := N_1(\phi(t))$. Then the geodesic curvatures of γ_1, γ_2 are related by*

$$\kappa_{\gamma_2}(t) = \kappa_{\gamma_1}(\phi(t)).$$

Proof. It is easy to see that $\phi(t) = \epsilon t + a$ for some constants $\epsilon = \pm 1$, $a \in \mathbb{R}$, so that

$$\gamma_2(t) = \gamma_1(\epsilon t + a).$$

Hence,

$$\dot{\gamma}_2(t) = \epsilon\dot{\gamma}_1(\epsilon t + a), \quad \ddot{\gamma}_2(t) = \ddot{\gamma}_1(\epsilon t + a).$$

This yields

$$\kappa_{\gamma_2}(t)N_2(t) = \frac{\nabla}{dt}\dot{\gamma}_2(t) = \left.\frac{\nabla}{ds}\right|_{s=\phi(t)} \dot{\gamma}_1(s) = \kappa_{\gamma_1}(\phi(t))N_1(\phi(t)),$$

from which the lemma follows. □

Corollary 18.1 *Let S be a regular surface and $R \subset S$ a smooth domain. There is a unique smooth function $\kappa_g : \partial R \to \mathbb{R}$ with the following property. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a smooth curve of unit speed such that $\gamma(t) \in \partial R$ for every t. If γ is given the inward-pointing normal orientation with respect to R then*

$$\kappa_g(\gamma(0)) = \kappa_\gamma(0).$$

Proof. Let γ_1, γ_2 be smooth curves of unit speed taking values on ∂R, both defined in open intervals containing 0. Then $\phi := \gamma_2^{-1} \circ \gamma_1$ is defined and smooth on a neighbourhood of 0. Now apply the lemma. $\square$

Let $R \subset S$ be a smooth region. Let $\gamma : \mathbb{R} \to S$ be a smooth, simply periodic curve of unit speed and period L such that ∂R equals the trace of γ. By the corollary, the integral

$$\int_{\partial R} \kappa_g \, ds := \int_0^L \kappa_\gamma(t) \, dt$$

will not depend on the choice of γ.

19 The local Gauss-Bonnet theorem, I

Theorem 19.1 *Let S be a regular surface with Gauss curvature K. Suppose $R \subset S$ is a smooth region which is contained in a chart domain for S. Then*

$$\int_R K \, dA + \int_{\partial R} \kappa_g \, ds = 2\pi.$$

Proof. Let $F : U \to S$ be a local parametrization with $R \subset F(U)$. Let X_1, X_2 be the corresponding coordinate vector fields and (E_1, E_2) the orthonormal frame on $F(U)$ obtained from (X_1, X_2) by the Gram-Schmidt process. We give $F(U)$ the orientation for which F is orientation preserving. Combining Proposition 14.1 and Stokes's theorem we find that

$$\int_R K \, dA = \int_R d\omega = \int_{\partial R} \omega, \tag{6}$$

where $\omega \in \Omega^1(F(U))$ is the connection form of the frame (E_1, E_2). To compute the line integral, choose a smooth, simply periodic curve $\gamma : \mathbb{R} \to S$ of unit speed whose trace equals ∂R. Let $N : \mathbb{R} \to S^2$ be the inward-pointing normal orientation of γ. By replacing $\gamma(t)$ by $\gamma(-t)$ if necessary, we can arrange that γ is positively oriented.

We can write

$$\dot\gamma(t) = \sum_i \beta^i(t) E_i(\gamma(t))$$

where each β^i is a smooth funtion $\mathbb{R} \to \mathbb{R}$. Then $\beta := (\beta^1, \beta^2)$ is a smooth curve in $\mathbb{R}^2 - \{(0, 0\}$. By Lemma 17.1 there is a smooth function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$\beta(t) = (\cos\theta(t), \sin\theta(t))$$

for all t. Then

$$\dot{\gamma}(t) = \cos\theta(t)E_1(\gamma(t)) + \sin\theta(t)E_2(\gamma(t)).$$

Furthermore the normal oriantation of γ is

$$N(t) = -\sin\theta(t)E_1(\gamma(t)) + \cos\theta(t)E_2(\gamma(t)),$$

as one can verify by computing $\mu_{\gamma(t)}(\dot{\gamma}(t), N(t)) = 1$, where μ is the volume form on S. Now,

$$\begin{aligned}\frac{\nabla}{dt}\dot{\gamma}(t) = &-\dot{\theta}(t)\sin\theta(t)E_1(\gamma(t)) + \cos\theta(t)\nabla_{\gamma(t),\dot{\gamma}(t)}E_1 \\ &+\dot{\theta}(t)\cos\theta(t)E_2(\gamma(t)) + \sin\theta(t)\nabla_{\gamma(t),\dot{\gamma}(t)}E_2.\end{aligned}$$

Inserting

$$\begin{aligned}\nabla_{\gamma(t),\dot{\gamma}(t)}E_1 &= -\omega_{\gamma(t)}(\dot{\gamma}(t))E_2(\gamma(t)), \\ \nabla_{\gamma(t),\dot{\gamma}(t)}E_2 &= \omega_{\gamma(t)}(\dot{\gamma}(t))E_1(\gamma(t)),\end{aligned}$$

we get

$$\frac{\nabla}{dt}\dot{\gamma}(t) = \left(\dot{\theta}(t) - \omega_{\gamma(t)}(\dot{\gamma}(t))\right)N(t).$$

Hence, the geodesic curvature of γ is

$$\kappa_\gamma(t) = \dot{\theta}(t) - \omega_{\gamma(t)}(\dot{\gamma}(t)). \tag{7}$$

If γ has period L then this yields

$$\int_{\partial R}\omega = \int_0^L \left(\dot{\theta}(t) - \kappa_\gamma(t)\right)dt = \theta(L) - \theta(0) - \int_{\partial R}\kappa_g ds.$$

Combining this with (6) we obtain

$$\int_R K\,dA + \int_{\partial R}\kappa_g\,ds = 2\pi W(\beta;0).$$

It only remains to prove that the winding number $W(\beta;0) = 1$. To this end we compare γ with the plane curve $\alpha := F^{-1}\circ\gamma$. Clearly, α is regular and simple periodic. Let $\alpha = (\alpha^1, \alpha^2)$. Since $\gamma = F\circ\alpha$, the chain rule yields

$$\dot{\gamma}(t) = \sum_i \dot{\alpha}^i(t)\partial_i F(\alpha(t)) = \sum_i \dot{\alpha}^i(t)X_i(\gamma(t)).$$

For fixed t, and omitting t and γ from notation for a moment, we then have

$$\dot{\gamma} = \sum_i \dot{\alpha}^i X_i = \sum_i \beta^i E_i.$$

Recall that the Gram-Schmidt process transforms a basis by a triangular matrix with positive entries on the diagonal. In our case,

$$E_j = \sum_i c^i_j X_i,$$

where the matrix (c^i_j) satisfies $c^i_i > 0$, and $c^i_j = 0$ for $i > j$. Therefore,

$$\dot{\gamma} = \sum_j \beta^j \sum_i c^i_j X_i = \sum_i \left(\sum_j c^i_j \beta^j \right) X_i.$$

This shows that

$$\dot{\alpha}^i = \sum_j c^i_j \beta^j.$$

Since the matrix (c^i_j) has only positive eigenvalues (namely c^i_i), we see that $\dot{\alpha}(t)$ is never a negative real multiple of $\beta(t)$. It is then a simple exercise to show that the curves $\dot{\alpha}$ and β have the same winding number with respect to the origin. Thus,

$$W(\beta; 0) = W(\dot{\alpha}; 0) = n_\alpha = 1,$$

where the last equality is the theorem of Hopf. This completes the proof of the theorem. □

As an example, let S^2 be the unit sphere in $\mathbb{R}^3$, and let $R \subset S^2$ be the upper hemisphere. Since ∂R is a great circle, which can be parametrized by a geodesic, we have $\kappa_g = 0$. Moreover, $K = 1$, so the theorem says that

$$2\pi = \int_R K\, dA + \int_{\partial R} \kappa_g\, ds = \int_R 1\, dA = \text{Area}(R) = \frac{1}{2}\text{Area}(S^2),$$

confirming that the area of S^2 is 4π.

20 The local Gauss-Bonnet theorem, II

Given a manifold M, a continuous curve $\gamma : I \to M$ is called **piecewise regular** if for all $a, b \in I$ with $a < b$ there exists a non-negative integer r and a partition

$$a = a_0 < a_1 < \cdots < a_r = b$$

such that the restriction of γ to the subinterval $[a_{i-1}, a_i]$ is a regular curve for $i = 1, \dots, r$.

Let S be a regular surface. By a **polygonal region** in S we mean a compact subset $R \subset S$ such that the following hold.

- R is the closure of an open subset of S.
- The boundary ∂R has finitely many components.
- Each component of ∂R is the image of a simple periodic, piecewise regular curve $\mathbb{R} \to S$.

A polygonal region R is called **simple** if R is contained in a chart domain for S and ∂R has exactly one boundary component.

Let $R \subset S$ be a polygonal region and $\gamma : \mathbb{R} \to S$ a simple periodic, piecewise regular curve of unit speed whose trace is a boundary component of R. If $t_0 \in \mathbb{R}$ is a point where γ is not smooth then $\gamma(t_0)$ is called a **vertex** of ∂R. A vertex $\gamma(t_0)$ is called a **cusp** if the one-sided derivatives $\dot\gamma(t_0^\pm)$ of γ at t_0 satisfy

$$\dot\gamma(t_0^+) = -\dot\gamma(t_0^-);$$

otherwise $\gamma(t_0)$ is called an **ordinary vertex**. If $\theta \in [0, 2\pi]$ is the interior angle of ∂R at a vertex p then $\epsilon := \pi - \theta \in [-\pi, \pi]$ is called the **jump angle** at p. If γ is smooth on a non-empty open interval (t_0, t_1) but not smooth at t_0 or at t_1 then the image of the closed interval $[t_0, t_1]$ under γ is called an **edge** of ∂R.

Let $J \subset \mathbb{R}$ be the largest open interval on which γ is smooth. Let $V := \gamma(\mathbb{R} - J)$ be the set of vertices in ∂R, which is finite. For the restriction of γ to J, the inward normal orientation and geodesic curvature can be defined as before, and we obtain a smooth function

$$\kappa_g : \partial R - V \to \mathbb{R}$$

characterized as in Corollary 18.1.

Theorem 20.1 *Let S be a regular surface with Gauss curvature K. Suppose $R \subset S$ is a simple polygonal region with jump angles $\epsilon_1, \dots, \epsilon_k$ at the vertices. Then*

$$\int_R K\,dA + \int_{\partial R} \kappa_g\,ds + \sum_{i=1}^{k} \epsilon_i = 2\pi.$$

Idea of proof. "Round off the corners" of ∂R to produce a smooth region $R' \subset S$ to which Theorem 19.1 can be applied. Use Equation (7) to estimate the integral $\int_{\partial R'} \kappa_g \, ds$. □

A simple polygonal region $R \subset S$ is called a **geodesic triangle** if R has exactly three vertices and each edge of ∂R can be parametrized by a geodesic.

Theorem 20.2 *Let $R \subset S$ be a geodesic triangle with interior angles θ_i, $i = 1, 2, 3$. Then*

$$\int_R K \, dA = \sum_{i=1}^{3} \theta_i - \pi.$$

Proof. The jump angle at the ith vertex is $\epsilon_i = \pi - \theta_i$. Since $\kappa_g = 0$, Theorem 20.1 gives

$$\int_R K \, dA = 2\pi - \sum_{i=1}^{3} (\pi - \theta_i) = \sum_{i=1}^{3} \theta_i - \pi. \qquad \square$$

If K is constant then the theorem says that

$$K \cdot \text{Area}(R) = \sum_{i=1}^{3} \theta_i - \pi.$$

Note that the cases $K = 0, 1, -1$ correspond to Euclidean, spherical, and hyperbolic triangles, respectively.

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