UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4510 — Geometric structures.

Day of examination: Monday, December 16, 2013.

Examination hours: 09.00 – 13.00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Each item (1a, 1b, 2a, etc.) counts 10 points.

Problem 1

a) Let $f \in \text{Möb}^+(\mathbb{H})$ be defined by

$$f(z) = \frac{2}{2 - z}.$$

Decide wether f is parabolic, hyperbolic or elliptic. Write f explicitly as a conjugate of a map on normal form.

b) Let $f \in \text{Möb}^-(\mathbb{H})$ be defined by

$$f(z) = \frac{4\bar{z} + 20}{5\bar{z} + 4}.$$

Find the fixpoints of f. Find an inversion $g \in \text{Möb}^-(\mathbb{H})$ and a hyperbolic transformation $k \in \text{Möb}^+(\mathbb{H})$ such that $f = g \circ k$ and $g \circ k = k \circ g$. g is an inversion in a $\mathbb{H}$-line ℓ, find ℓ.

Problem 2

a) Find the hyperbolic area of the Euclidean triangle in $\mathbb{H}$ with vertices $i, i + 1$ and $2i$.

b) Consider a hyperbolic triangle with the angles α, β and γ and opposite sides of hyperbolic length a, b and c respectively. Assume $b = c$. Show that $\beta = \gamma$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the $\mathbb{H}$-lines $|z + 1| = \sqrt{2}$ and $|z - 1| = \sqrt{2}$, $z \in \mathbb{H}$ respectively. Let $z_1 \in \mathcal{C}_1$ and $z_2 \in \mathcal{C}_2$ be points such that $d_{\mathbb{H}}(z_1, i) = d_{\mathbb{H}}(z_2, i) = \ln(\sqrt{2} + \sqrt{3})$ and such that $\text{Re } z_1 < 0$ and $\text{Re } z_2 > 0$. Let T be the hyperbolic triangle with vertices z_1, i and z_2.

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c) Calculate $\cosh(d_{\mathbb{H}}(z_1, z_2))$. (Hint: Apply the first law of cosine to T.)

d) Calculate the hyperbolic area of T.

Problem 3

A regular surface $S \subset \mathbb{R}^3$ is parametrized by

$$\mathbf{x}(u, v) = (u, v, uv),\ u, v \in \mathbb{R}.$$

Another regular surface S' is parametrized by

$$\mathbf{y}(u, v) = (\cosh v \cos u, \cosh v \sin u, \sinh v),\ u, v \in \mathbb{R}.$$

a) Find the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ of S and of S' with respect to these parametrizations.

b) Find the Gaussian curvature K_S and $K_{S'}$ of S and S' respectively. Does there exists an open neighborhood U of $(0, 0, 0) \in S$ which is isometric to some open set $V \subset S'$?

c) Show that the curve given by $\alpha(t) = (t, t, t^2)$ is a geodesic on S. Also show that the curves $\beta(t) = (t, v_0, v_0 t)$, and the curves $\gamma(t) = (u_0, t, u_0 t)$ (where u_0 and v_0 are constants) are geodesics on S.

d) Let $T_a = \{(u, v) \,|\, u \in [0, a], 0 \leq v \leq u\}$. Let R_a be the region in S parametrized over T_a. Let η_1, η_2 and η_3 be the interior angles of R_a at the points $(0, 0, 0)$, $(a, 0, 0)$ and (a, a, a^2) respectively. Show that $\eta_1 = \frac{\pi}{4}$, $\eta_2 = \frac{\pi}{2}$ and that $\cos \eta_3 = \sqrt{\frac{1+2a^2}{2+2a^2}}$. Apply The Gauss-Bonnet Theorem to R_a to show that

$$\lim_{a\to\infty} \int\!\!\int_{T_a} \frac{dudv}{(1 + u^2 + v^2)^{3/2}} = \frac{\pi}{4}.$$

THE END