MANDATORY ASSIGNMENT FOR MAT4520 SPRING 2013

JOHN ROGNES

Return to John Rognes by Wednesday April 24th. Each of the 15 questions 1(a,b,c) and 2(a,...,l) carry equal weight. A score over 50% is sufficient to pass. A near-pass may qualify for a second attempt. You may cooperate with other students, but your answers should reflect your own understanding.

Problem 1 (20%)

Same as Spivak, Chapter 3, Problem 33(a), (b) and (c). (Hint: Use the "trivial case" of Sard's theorem.)

Problem 2 (80%)

Let $M(3) = \{A = (a_{ij})_{i,j}\} \cong \mathbb{R}^9$ be the vector space of 3×3 real matrices, with the usual smooth structure. Let $A^T = (a_{ji})_{i,j}$ denote the transpose of A, and let $\mathrm{Sym}(3) = \{C \in M(3) \mid C^T = C\} \cong \mathbb{R}^6$ and $\mathrm{Skew}(3) = \{C \in M(3) \mid C^T = -C\} \cong \mathbb{R}^3$ be the subspaces of symmetric and skew-symmetric matrices, respectively. Let $f\colon M(3) \to \mathrm{Sym}(3)$ be the smooth map given by $f(A) = A \cdot A^T$, let I be the identity matrix, and let $O(3) = \{A \in M(3) \mid A \cdot A^T = I\}$ be the subspace of orthogonal matrices.

(a) Show that $f_{*A}\colon T_A M(3) \to T_{f(A)} \mathrm{Sym}(3)$ is given by

$$f_{*A}(C) = A \cdot C^T + C \cdot A^T \,.$$

(Hint: Consider the image under f of a curve $t \mapsto A + tC$ in $M(3)$ through A with tangent vector C.)

(b) Show that I is a regular value of f. (Hint: If $A \cdot C^T = \frac{1}{2}D$ then $C \cdot A^T = \frac{1}{2}D^T$.)

(c) Explain why $O(3)$ is a smooth submanifold of $M(3)$, and determine its dimension. (Hint: Cite a result from the book.)

Let $k\colon O(3) \hookrightarrow M(3)$ denote the inclusion map.

(d) Prove that the image of $k_{*A}\colon T_A O(3) \hookrightarrow T_A M(3) \cong M(3)$ is equal to the subspace

$$\{C \in M(3) \mid A \cdot C^T + C \cdot A^T = 0\}\,.$$

(Hint: Prove one inclusion, then compare dimensions.)

Hereafter we use k_{*A} to identify $T_A O(3)$ with this subspace of $M(3)$. In particular, k_{*I} identifies $T_I O(3)$ with $\mathrm{Skew}(3)$. For each $C \in \mathrm{Skew}(3)$ let $X(C)\colon O(3) \to TO(3)$ be the smooth vector field with $X(C)_A = A \cdot C$ in $T_A O(3)$, for each $A \in O(3)$.

(e) Show that $X = X(C)$ is left invariant, in the sense that $\ell_* X = X$ for each diffeomorphism $\ell\colon O(3) \to O(3)$ of the form $\ell(A) = B \cdot A$, where $B \in O(3)$.

Take as known the fact that the series

$$e^C = \sum_{n\geq 0} \frac{C^n}{n!} = I + C + \frac{C^2}{2} + \dots$$

defines a smooth map $e\colon M(3) \to M(3)$, which satisfies $e^{(s+t)C} = e^{sC} \cdot e^{tC}$ for $s, t \in \mathbb{R}$ and $C \in M(3)$. (See Spivak, Chapter 5, Problem 6 for more details.)

(f) Prove that e restricts to a smooth map $\exp\colon \mathrm{Skew}(3) \to O(3)$, and compute the derivative at $t = 0$ of the curve $t \mapsto \exp(tC)$ through I in $O(3)$.

(g) Let $C \in \mathrm{Skew}(3)$ and let $X = X(C)$ be the associated left invariant vector field with $X_I = C$. Show that the 1-parameter group of diffeomorphisms $\{\phi_t\}_t$ generated by X is given by

$$\phi_t(A) = A \cdot \exp(tC)$$

for $t \in \mathbb{R}$ and $A \in O(3)$.

(h) Let $D \in \mathrm{Skew}(3)$ and let $Y = X(D)$ be the associated left invariant vector field with $Y_I = D$. Show that

$$(\phi_t)_*(Y)_A = A \cdot \exp(tC)^{-1} \cdot D \cdot \exp(tC)$$

for all $t \in \mathbb{R}$ and $A \in O(3)$.

(i) Prove that the Lie derivative $L_X Y$ of $Y = X(D)$ with respect to $X = X(C)$ satisfies

$$L_{X(C)} X(D) = X([C, D])$$

for $C, D \in \mathrm{Skew}(3)$, where the commutator $[C, D] = C \cdot D - D \cdot C$ is computed in $\mathrm{Skew}(3) \subset M(3)$. (Hint: Use the definition of the Lie derivative to prove that $(L_X Y)_A = A \cdot (C \cdot D - D \cdot C)$ for all $A \in O(3)$.)

Let $\xi\colon O(3) \to \mathbb{R}^3$ be given by $\xi(A) = (a_{23}, a_{13}, a_{12})$, where $A = (a_{ij})_{i,j}$, so that $\xi(I) = 0 = (0, 0, 0)$.

(j) Show that $\xi_{*I}\colon T_I O(3) \to T_0\mathbb{R}^3 \cong \mathbb{R}^3$ is an isomorphism. Explain why there is an open neighborhood $U \subset O(3)$ of I, with $\xi(U) \subset \mathbb{R}^3$ an open neighborhood of 0, such that $\xi|U\colon U \to \xi(U)$ is a diffeomorphism.

Let $x = \xi|U$, so that (x, U) is a smooth chart on $O(3)$ near I.

(k) Compute the images C, D and $E \in \mathrm{Skew}(3)$ (under k_{*I}) of the vectors

$$\frac{\partial}{\partial x^1}\Big|_I \,,\ \frac{\partial}{\partial x^2}\Big|_I \text{ and } \frac{\partial}{\partial x^3}\Big|_I \ \in\ T_I O(3)\,,$$

respectively.

(l) Compute the commutators $[C, D]$, $[D, E]$ and $[E, C]$ in $\mathrm{Skew}(3)$. Determine the Lie derivatives $L_{X(C)} X(D)$, $L_{X(D)} X(E)$ and $L_{X(E)} X(C)$.

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