SAMPLE SOLUTION FOR MAT4520 SPRING 2013

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Problem 1 (20%)

(a) Let $U = \{(p,q) \in M \times M| \mid p \neq q\}$ and $V = \{(p,w) \in TM \mid w \neq 0\}$ be open submanifolds of $M \times M$ and TM. Let $f\colon U \sqcup V \to S^{N-1}$ be given by $f(p,q) = (p-q)/|p-q|$ and $f(p,w) = w/|w|$. Then $U \sqcup V$ is a smooth $2n$-manifold with at most countably many components, and $f\colon U \sqcup V \to S^{N-1}$ is smooth, so the set of critical values of f has measure zero. Since $2n < N-1$ this is the same as the set of values of f, so the image of f has measure zero, so we can find a vector $v \in S^{N-1}$ that is not in the image of f. Then no chord of M is parallel to v, and no tangent space T_pM contains v.

(b) The map $\pi|M\colon M \to \mathbb{R}^{N-1}$ is one-to-one because if $p, q \in M$ satisfy $\pi(p) = \pi(q)$ then $p-q$ is parallel to v, and this does not happen for $p \neq q$. The differential $(\pi|M)_{*p}\colon T_pM \to \mathbb{R}^{N-1}$ equals the composite of the inclusion $T_pM \to T_p\mathbb{R}^N \cong \mathbb{R}^N$ followed by the projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^{N-1}$. It is injective because if $(\pi|M)_{*p}(w) = 0$ then $\pi(w) = 0$, so $w \in T_pM$ is a non-zero multiple of v, which is impossible since the tangent space T_pM does not contain v.

(c) If M is a compact smooth n-manifold, there exists an embedding $M \hookrightarrow \mathbb{R}^N$ for some N, by Spivak, Chapter 2, Theorem 17. If $N > 2n+1$ we can find a projection $\pi\colon \mathbb{R}^N \to \mathbb{R}^{N-1}$ as above, such that $\pi|M$ is an embedding $M \hookrightarrow \mathbb{R}^{N-1}$. By descending induction, we can repeat until $N = 2n+2$, when we get an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.

Problem 2 (80%)

(a) Let $c(t) = A + tC$ be a curve through $c(0) = A$ with tangent vector $c'(0) = C$. Then $f_{*A}(C)$ is the tangent vector of

$$fc(t) = (A+tC)\cdot(A+tC)^T = A\cdot A^T + t(A\cdot C^T + C\cdot A^T) + t^2 C\cdot C^T$$

at $t = 0$, which is $(fc)'(0) = A\cdot C^T + C\cdot A^T$.

(b) We must prove that $f_{*A}\colon T_AM(3) \to T_I\,\mathrm{Sym}(3) \cong \mathrm{Sym}(3)$ is surjective for each $A \in O(3)$. Consider any $D \in \mathrm{Sym}(3)$. Let $C = \frac{1}{2}D\cdot A$. Then $A\cdot C^T + C\cdot A^T = \frac{1}{2}D^T + \frac{1}{2}D = D$, since D is symmetric. Hence $f_{*A}(C) = D$, so f_{*A} is surjective.

(c) I is a regular value of $f\colon M(3) \to \mathrm{Sym}(3)$, so $O(3) = f^{-1}(I)$ is a submanifold of $M(3)$ of dimension $n - m = 9 - 6 = 3$, where $n = \dim M(3) = 9$ and $m = \dim \mathrm{Sym}(3) = 6$, by Spivak, Chapter 2, Proposition 12.

(d) The composite $fk\colon O(3) \to \mathrm{Sym}(3)$ is constant at I, so $0 = (fk)_{*A}(v) = f_{*A}(k_{*A}(v))$ for any tangent vector $v \in T_AO(3)$. Hence $C = k_{*A}(v)$ satisfies $0 = f_{*A}(C) = A\cdot C^T + C\cdot A^T$. Thus the image of k_{*A} is contained in the subspace $\{C \in M(3) \mid A\cdot C^T + C\cdot A^T = 0\}$.

The map $D \mapsto D \cdot A$ takes $\mathrm{Skew}(3) \cong \mathbb{R}^3$ isomorphically to this subspace, since $A \cdot (D \cdot A)^T + (D \cdot A) \cdot A^T = D^T + D$ equals 0 if and only if D is skew-symmetric. Hence the image of $T_A O(3)$ in $\{C \in M(3) \mid A \cdot C^T + C \cdot A^T = 0\}$ is a 3-dimensional subspace in a 3-dimensional vector space, which implies that they are equal.

(e) The diffeomorphism $\ell \colon O(3) \to O(3)$ is the restriction of the linear map $l \colon M(3) \to M(3)$ given by $l(A) = B \cdot A$, so $l_{*A} \colon T_A M(3) \to T_{l(A)} M(3)$ is also the linear map given by $l_{*A}(C) = B \cdot C$. Hence $\ell_{*A} \colon T_A O(3) \to T_{\ell(A)} O(3)$ is also given by $\ell_{*A}(C) = B \cdot C$, under the identification of $T_A O(3)$ with a subspace of $T_A M(3) \cong M(3)$. Then

$$\begin{aligned}(\ell_* X)_A &= \ell_{*\ell^{-1}(A)}(X_{\ell^{-1}(A)}) \\ &= \ell_{*B^{-1}\cdot A}(B^{-1} \cdot A \cdot C) = B \cdot B^{-1} \cdot A \cdot C = A \cdot C = X_A\end{aligned}$$

for each $A \in O(3)$, so $\ell_* X = X$.

(f) It suffices to prove that $e|\,\mathrm{Skew}(3)$ factors through the submanifold $O(3) \subset M(3)$, i.e., to show that $e(C) \cdot e(C)^T = I$ for $C \in \mathrm{Skew}(3)$. Note that $e(C)^T = e(C^T)$ from the series definition. Then $e(C)^T = e(C^T) = e(-C)$, so $e(C) \cdot e(C)^T = e(C) \cdot e(-C) = e((1-1)C) = e(0C) = I$, as required. The derivative of $t \mapsto \exp(tC)$ at $t = 0$ is

$$\lim_{t \to 0} \frac{\exp(tC) - \exp(0C)}{t} = \lim_{t \to 0} \sum_{n \geq 1} t^{n-1} \frac{C^n}{n!} = C$$

in $T_I O(3) \cong \mathrm{Skew}(3)$.

(g) The map is well defined, because the product $A \cdot \exp(tC)$ of two orthogonal matrices is again orthogonal. We check that $\{\phi_t\}_t$ satisfies Spivak, Chapter 5, Theorem 6, i.e., that it satisfies the three conditions of Theorem 5 for all $t \in \mathbb{R}$.

(1) The map $\mathbb{R} \times O(3) \to O(3)$ given by $(t, A) \mapsto \phi_t(A) = A \cdot \exp(tC)$ is smooth, since exp and matrix multiplication is smooth.

(2) We have $\phi_{s+t}(A) = \phi_{t+s}(A) = A \cdot \exp((t+s)C) = A \cdot \exp(tC) \cdot \exp(sC) = \phi_s(\phi_t(A))$ for all $s, t \in \mathbb{R}$ and $A \in O(3)$.

(3) The tangent vector of the curve $t \mapsto c(t) = \phi_t(A) = A \cdot \exp(tC)$ at $t = 0$ is $c'(0) = A \cdot C$, by the previous exercise (and the Leibniz rule), and this equals X_A at each point $A \in O(3)$.

(h) The diffeomorphism $\phi_t \colon O(3) \to O(3)$ is the restriction of the linear map $p \colon M(3) \to M(3)$ given by $p(A) = A \cdot \exp(tC)$, so $p_{*A} \colon T_A M(3) \to T_{p(A)} M(3)$ is also the linear map given by $p_{*A}(D) = D \cdot \exp(tC)$. Hence $(\phi_t)_{*A} \colon T_A O(3) \to T_A O(3)$ is also given by $(\phi_t)_{*A}(D) = D \cdot \exp(tC)$. Then

$$\begin{aligned}(\phi_t)_*(Y)_A &= (\phi_t)_{*\phi_t^{-1}(A)}(Y_{\phi_t^{-1}(A)}) \\ &= (\phi_t)_{*A\cdot\exp(tC)^{-1}}(A \cdot \exp(tC)^{-1} \cdot D) = A \cdot \exp(tC)^{-1} \cdot D \cdot \exp(tC)\,.\end{aligned}$$

(i) For each $A \in O(3)$, the Lie derivative

$$\begin{aligned}(L_X Y)_A &= \lim_{h \to 0} \frac{1}{h}(Y_A - \phi_{h*}(Y)_A) \\ &= \lim_{h \to 0} \frac{1}{h}(A \cdot D - A \cdot \exp(hC)^{-1} \cdot D \cdot \exp(hC)) \\ &= -\lim_{h \to 0} \frac{1}{h}(A \cdot \exp(hC)^{-1} \cdot D \cdot \exp(hC) - A \cdot D)\end{aligned}$$

(a limit formed in $T_A O(3) \subset T_A M(3) \cong M(3)$) equals $-c'(0)$, where

$$c(t) = A \cdot \exp(tC)^{-1} \cdot D \cdot \exp(tC) = A \cdot \exp(-tC) \cdot D \cdot \exp(tC)\,.$$

By the Leibniz rule and part (f), $c'(0) = A\cdot(-C)\cdot D + A\cdot D\cdot C = A\cdot(-C\cdot D + D\cdot C)$, so $(L_X Y)_A = -c'(0) = A\cdot(C\cdot D - D\cdot C) = A\cdot[C,D] = X([C,D])_A$. Hence the vector field $L_{X(C)}X(D) = L_X Y$ equals $X([C,D])$.

(j) The map ξ is the restriction of the linear map $M(3) \to \mathbb{R}^3$ taking A to (a_{23}, a_{13}, a_{12}), hence ξ_{*I} is the restriction to $\mathrm{Skew}(3) \cong T_I O(3) \subset T_I M(3) \cong M(3)$ of the same linear map. The matrices

$$C' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \; D' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \text{ and } E' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

form a basis for $\mathrm{Skew}(3)$, and are mapped to the standard basis vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ for $\mathbb{R}^3$ by ξ_{*I}, hence ξ_{*I} is an isomorphism.

By the Inverse Function Theorem (or Spivak, Chapter 2, Theorem 9), ξ restricts to a diffeomorphism on an open neighborhood U of I.

(k) The vector $\partial/\partial x^i|_I$ in $T_I U \cong T_I O(3)$ maps under $x_{*I} = \xi_{*I}$ to the i-th standard basis vector e_i in $T_0\mathbb{R}^3 \cong \mathbb{R}^3$. The matrices C', D' and E' in $\mathrm{Skew}(3)$ displayed above have this property, for $i = 1$, 2 and 3, respectively. Since $x_{*I} = \xi_{*I}$ is an isomorphism, this characterizes these vectors, so $C = C'$, $D = D'$ and $E = E'$.

(l)

$$C \cdot D = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (D \cdot C)^T$$

$$D \cdot E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = (E \cdot D)^T$$

$$E \cdot C = \begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (C \cdot E)^T$$

so $[C,D] = E$, $[D,E] = C$ and $[E,C] = D$. Hence $L_{X(C)}X(D) = X(E)$, $L_{X(D)}X(E) = X(C)$ and $L_{X(E)}X(C) = X(D)$.

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