EXERCISES CHAPTER 7

§7 - Quotient spaces

Let M be a topological n-dimensional manifold and let G be a group. Assume that G is a continuous left action on M, i.e., for each $g \in G$ there is associated an element $gx \in M$ for all $x \in M$, the map $g : M \to M$ is continuous, $ex = x$ for all $x \in M$ where $e \in G$ denotes the identity element, and $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in G$ and $x \in M$.

(a) Prove that for each $g \in G$ the map $g : M \to M$ is a bijection.
(b) Prove that for each $g \in G$ the map $g : M \to M$ is a homeomorphism.
(c) For $x, y \in M$ set $x \sim y$ if $y = gx$ for some $g \in G$. Prove that $\sim$ is an equivalence relation.
(d) Prove that the quotient map $\pi : M \to M/\sim$ is an open map.

Assume now that G is a finite group without fixed points, i.e., if $x \in M$ and $g \in G$ satisfies $gx = x$ then $g = e$.

(f) Prove that $M/\sim$ is a Hausdorff space.
(g) Prove that $M/\sim$ is a topological manifold.
(h) Assume that for each $g \in G$ we have that $g : M \to M$ is a diffeomorphism. Prove that $M/\sim$ has a natural structure as a smooth manifold.

Example 0.1. Let $M = S^n$. And let G be the group generated by the map $gx = -x$. Then $g^2 = e$ and so $G \approx \mathbb{Z}_2$. One can check that $g : M \to M$ is a diffeomorphism (check it!). So $M/\sim$ has a natural structure as a smooth manifold. We have seen that $M/\sim$ is homeomorphic to the real projective plane $\mathbb{RP}^n$.