MAT4520 - Some hints and suggested solutions

January 2021

§3

3.1 See end of book.

3.4 Suppose that f is alternating. If $\sigma \in S_k$ simply switches two elements and leave the rest fixed, then $\text{sgn}(\sigma) = -1$ so the claim follows. Next suppose that $\sigma f = -f$ whenever σ simply switches two successive elements and leave the rest fixed. Every $\sigma \in S_n$ may be written as a product $\sigma = \sigma_m \cdot ... \cdot \sigma_1$ of such permutations (prove it!). We then see that

$$\sigma f = (-1)^m f = \text{sgn}(\sigma_m) \cdot ... \cdot \text{sgn}(\sigma_1) f = \text{sgn}(\sigma) f.$$

3.5 Suppose that f is alternating. Then if $i < j$ and if $v_i = v_j$ we have that

$$f(v_1, .., v_i, ..., v_j, ..., v_k) = -f(v_1, ..., v_j, ..., v_i, ..., v_k) = -f(v_1, .., v_i, ..., v_j, ..., v_k),$$

so $f(v_1, .., v_i, ..., v_j, ..., v_k) = 0$.

Suppose next that f satisfies the equality condition. Then

$$\begin{aligned} 0 &= f(u_1, ..., u_i + u_j, ..., u_j + u_i, ..., u_k) \\ &= f(u_1, ..., u_i, ..., u_j, ..., u_k) + f(u_1, ..., u_i, ..., u_i, ..., u_k) \\ &+ f(u_1, ..., u_j, ..., u_j, ..., u_k) + f(u_1, ..., u_j, ..., u_i, ..., u_k) \\ &= f(u_1, ..., u_i, ..., u_j, ..., u_k) + f(u_1, ..., u_j, ..., u_i, ..., u_k) \end{aligned}$$

Now use **3.4**.

§4

§5

5.1 (a) Consider the topological space X consisting of two disjoint copies of $\mathbb{R}$, and denote them by $\mathbb{R}_1$ and $\mathbb{R}_2$. Define a map $p : X \to S$ as follows: on $\mathbb{R}_1 \setminus \{0\}$ we set $p(x) = x$ and for $0 \in \mathbb{R}_1$ we set $p(0) = A$; on $\mathbb{R}_2 \setminus \{0\}$ we set $p(x) = x$ and for $0 \in \mathbb{R}_2$ we set $p(0) = B$. Then the quotient topology on S coincides with the topology given in the problem.

Consider an interval $(a, b)_1 \subset \mathbb{R}_1$. Then if $b \leq 0$ we have that

$$p^{-1}(p(a, b)) = (a, b)_1 \cup (a, b)_2$$

which is open in X. Analogously, if $0 \leq a$ we have $p^{-1}(p(a, b)) = (a, b)_1 \cup (a, b)_2$ which is is open. If $a < 0 < b$ we have that

$$p^{-1}(p(a, b)) = (a, 0)_1 \cup \{A\} \cup (0, b)_1 \cup (a, b)_2 \setminus \{0\}$$

which is open in X. It follows that $p : \mathbb{R}_1 \to S$ is a bijective continuous open map onto its image, i.e., it is a homeomorphism onto its image. Similar considerations hold for $\mathbb{R}_2$. It is now immediate that h is a homeomorphism and that S is locally euclidean.

(b) By (a) we have that $p : X \to S$ is surjective continuous and open map; it follows that S is second countable (see lecure/note about quotient maps). However, we have that S is not Hausdorff because if $(a, 0) \cup \{A\} \cup (0, b)$ and $(a', 0) \cup \{B\} \cup (0, b')$ are open sets containing A and B respectively, and their intersection is $(a, 0) \cap (a', 0) \cup (0, b) \cap (0, b')$ which is always non-empty.

5.3 We have that ϕ_4 maps U_4 onto the open unit disk $\{(x, z) : x^2 + z^2 < 1\}$ in the (x, z)-plane. With the additional requirement that $x > 0$ we have that ϕ_4 maps U_{14} onto the half disk $\{(x, z) : x^2 + z^2 < 1, x > 0\}$.

We have that $\phi_4^{-1}(x, z) = (x, -\sqrt{1 - x^2 - z^2}, z)$, and so

$$\phi_1 \circ \phi_4^{-1}(x, z) = (-\sqrt{1 - x^2 - z^2}, z)$$

which is smooth since $\sqrt{1 - x^2 - z^2} \neq 0$ on $\phi_4(U_{14})$.

5.4 By perhaps having to chose a connected component of U we may assume that U is an n-dimensional (sub) manifold (of M). Since (U_α, ϕ_α) is an atlas there is a $\{(U_\alpha, \phi_\alpha)\}$ such that $p \in U_\alpha$. Set $V = U_\alpha \cap U$ and $\psi = \phi_\alpha|_V$. Then $\psi : V \to \psi(V)$ is a homeomorphism since ψ is the restriction of a homeomorphism to an open set. We now claim that (V, ψ) is compatible with the atlas. For if (U_β, ϕ_β) is in the atlas we have that

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n$$

is smooth. But now $\psi \circ \phi_\beta^{-1}$ is just the restriction of $\phi_\alpha \circ \phi_\beta^{-1}$ to $\phi_\beta(V \cap U_\beta)$ so it is smooth. A similar argument shows that $\phi_\beta \circ \psi^{-1}$ is smooth. This shows that (V, ψ) is compatible with the atlas, but then it is contained in the atlas since the atlas is maximal.

§6

6.1 (a) If we had that $(\mathbb{R}, \psi)$ were in the maximal standard atlas for $\mathbb{R}$, then the map $\psi \circ \phi^{-1}$ would be a diffeomorphism. But $\psi \circ \phi^{-1}(x) = x^{1/3}$ which is not differentiable at 0.

(b) Set $f(x) = x^3$. Then $\psi \circ f \circ \phi^{-1}(x) = (x^3)^{1/3} = x$. Likewise $\phi \circ f \circ \psi^{-1}(x) = (x^3)^{1/3} = x$. So f is a diffeomorphism.

6.2 Fix a point $p \in M$, and let (U, ϕ), (V, ψ) be charts around p and q_0 respectively. Then

$$(\phi \times \psi) \circ i_{q_0} \circ \phi^{-1}(x) = (x, \psi(q_0))$$

which is a smooth map between Euclidean spaces.

6.4 Set $\phi(x, y, z) = (x, x^2+y^2+z^2-1, z)$. We have that the rows of $J\phi(x, y, z)$ are the vectors $v_1(x) = (1, 0, 0)$, $v_2(x) = (2x, 2y, 2z)$ and $v_3 = (0, 0, 1)$. We see that the rank of $J\phi$ is three if and only if $y \neq 0$, so ϕ can serve as a coordinate system near all points (x, y, z) such that $y \neq 0$.

§7

(a) For each $x \in M$ we have that $g(g^{-1}(x)) = (gg^{-1})(x) = e(x) = x$, so g is surjective. If $g(x_1) = g(x_2)$ then $g^{-1}(g(x_1)) = g^{-1}(g(x_2))$ so $x_1 = x_2$, and g is injective.

(b) Since g is a continuous bijection it suffices to show that g is an open map. So let $U \subset M$ be open. Then $g(U) = (g^{-1})^{-1}(U)$ is open since g^{-1} is continuous.

(c) For $x \in M$ we have that $ex = x$ so $x \sim x$. If $x \sim y$ then $gx = y$ for some $g \in G$, and then $g^{-1}y = x$, so $y \sim x$. If $x \sim y$ and $y \sim z$ then $g_1 x = y$ and $g_2 y = z$ for $g_1, g_2 \in G$, and then $z = g_2(g_1 x) = (g_1 g_2)x$, so $x \sim z$.

(d) Let $U \subset M$ be an open set. Then

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU,$$

which is a union of open sets since gU is open, g being a homeomorphism.

(f) Let $G = \{g_0, ..., g_m\}$, where $g_0 = e$. For distinct points $[x_0], [y_0] \in M/\sim$ we set $x_j = g_j(x_0)$ and $y_j = g_j(y_0)$; these are then $2m + 2$ distinct points. Since M is Hausdorff there are open sets U_j and V_j containing x_j and y_j respectively, such that $U_i \cap U_j = \emptyset, V_i \cap V_j = \emptyset$ if $i \neq j$ and $U_i \cap V_j = \emptyset$

for all i, j. Set $\tilde{U}_0 = \cap_j g_j^{-1}(U_j)$ and $\tilde{V}_0 = \cap_j g_j^{-1}(V_j)$, and $\tilde{U}_j = g_j(\tilde{U}_0)$ and $\tilde{V}_j = g_j(\tilde{V}_0)$. Then $\tilde{U}_i \cap \tilde{V}_j = \emptyset$ for all i, j, and $\pi : \tilde{U}_i \to \pi(\tilde{U}_i)$ and $\pi : \tilde{V}_j \to \pi(\tilde{V}_j)$ are homeomorphisms onto disjoint open subsets separating $[x_0]$ and $[y_0]$ (injective and separating by construction, and open by (d)).

(g) For each point $[x_0]$ in $M/\sim$ Let $\tilde{U}_0$ be a set as constructed as above. For any coordinate chart (W, ϕ) near x_0 we use the homeomorphism

$$\phi \circ \pi^{-1} : \pi(W \cap \tilde{U}_0) \to \phi \circ \pi^{-1}(W \cap \tilde{U}_0)$$

where π^{-1} is the unique left inverse of π with image in $\tilde{U}_0$, as a coordinate chart near $[x_0]$. Similarly we obtain charts by considering $x_j \in \tilde{U}_j$ and charts at x_j. The charts are compatible since compositions of homeomorphisms are homeomorphisms (see also (h)).

(h) If $[x]$ is a point in coordinate charts $\pi(U) \cap \pi(V)$ where (U, ϕ) and (V, ψ) are charts, then near $\phi(x) \in \mathbb{R}^n$ the transition map is given by $\psi \circ g \circ \phi^{-1}$ for some $g \in G$ which is smooth since g is smooth.

§8

8.1. By Proposition 8.11 we have that

$$\begin{aligned} F_* \frac{\partial}{\partial x}|_p &= \frac{\partial f_1}{\partial x}(p) \frac{\partial}{\partial u}|_{F(p)} + \frac{\partial f_2}{\partial x}(p) \frac{\partial}{\partial v}|_{F(p)} + \frac{\partial f_3}{\partial x}(p) \frac{\partial}{\partial w}|_{F(p)} \\ &= \frac{\partial}{\partial u}|_{F(p)} + y \frac{\partial}{\partial w}|_{F(p)}. \end{aligned}$$

8.2. Fix a derivation $X_p = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}|_p$. Then if we set $c(t) = p + t \cdot a$ with $a = (a_1, ..., a_n)$ we have that $X_p = c'(0)$. Furthermore, if we set $b(t) = L(c(t))$ we have that $L_{*,p} X_p = b'(0)$, and we have seen $b'(0) = \sum_{j=0}^n \dot{b}_j(0) \frac{\partial}{\partial x_j}|_{b(0)}$. Finally $\dot{b}(0) = \frac{d}{dt}|_{t=0} L(c(0)) = L(\dot{c}(0))$ by the chain rule and the fact that L is linear; hence it coincides with its derivative.

8.4. Define $F(r, \theta) = (r \cos \theta, r \sin \theta) = (x, y)$. Then

$$\begin{aligned} F_{*,(r,\theta)} \frac{\partial}{\partial r}|_{(r,\theta)} &= \cos(\theta) \frac{\partial}{\partial x}|_{F(r,\theta)} + \sin(\theta) \frac{\partial}{\partial y}|_{F(r,\theta)} \\ &= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x}|_{(x,y)} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y}|_{(x,y)}. \end{aligned}$$

Further

$$F_{*,(r,\theta)}\frac{\partial}{\partial\theta}|_{(r,\theta)} = -r\sin\theta\frac{\partial}{\partial x}|_{F(r,\theta)} + r\cos\theta\frac{\partial}{\partial y}|_{F(r,\theta)}$$
$$= x\frac{\partial}{\partial y}|_{(x,y)} - y\frac{\partial}{\partial x}|_{(x,y)}$$

8.7. See end of book.

8.8. See end of book.

§9

9.1 See end of book.

9.2 See end of book.

9.3 See end of book.

9.4 See end of book.

9.5 See end of book.

§11

11.1 For a point $p \in S^n$ and a vector $a \in T_p\mathbb{R}^{n+1}$ we have that $a \in T_pS^n$ if and only if there is a smooth curve $c : (-\epsilon, \epsilon) \to S^n$ such that $c(0) = p$ and $c'(0) = a$ (or is represented by the vector a). Set $\rho(x) = \sum_{j=1}^{n+1} x_j^2$. Then $\rho(c(t)) \equiv 1$ and so

$$\frac{d}{dt}|_{t=0}\rho(c(t)) = 2 \cdot p \cdot a = 0,$$

here $\cdot$ is the dot-product. So T_pS^n is a subspace of

$$\text{Ker}\{a \mapsto p \cdot a\}.$$

Since the dimension of this kernel is n and the dimension of T_pS^n is n they must coincide.

11.3 See end of book.

11.6
(i) Write $g(A) = \det(A) - 1$; then $\frac{\partial g}{\partial x_{kl}}(A) \neq 0$. Replacing x_{kl} by g we get by Lemma 9.10 a coordinate chart (U, ϕ) near A such that

$$\phi(U \cap SL(n, \mathbb{R})) = \{x \in \phi(U) : x_{kl} = 0\}.$$

This is an adapted chart, and it follows that $(x_{ij})_{(i,j)\neq(k,l)}$ is a coordinate chart near A. The implicit function theorem says precisely that there is an open subset $U \subset \mathbb{R}^{n^2-1}$ and a smooth function h on U such that

$$\{x_{kl} = h((x_{ij}))_{(i,j)\neq(k,l)}, (x_{ij}) \in U\}$$

is precisely the set of points near A such that $g = 0$.

(ii) Assume for simplicity that $n = 2$, and consider a point $(A, B) \in SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Assume further for example that near A we have $(k, l) = (1, 1)$, near B we have $(k, l) = (1, 2)$ and near AB we have $(k, l) = (2, 1)$. Set $x' = (x_{12}, x_{21}, x_{22})$ and $x'' = (x_{11}, x_{21}, x_{22})$ Then $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is parametrized near (A, B) by

$$\begin{bmatrix} h_A(x') & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \times \begin{bmatrix} x_{11} & h_B(x'') \\ x_{21} & x_{22} \end{bmatrix}$$

Then we see that in the local coordinates near (A, B) and AB in $\mathbb{R}^3 \times \mathbb{R}^3$ and $\mathbb{R}^3$ respectively, we have that the matrix multiplication is given by

$$(h_A(x') \cdot x_{11} + x_{12} \cdot x_{21}, h_A(x') \cdot h_B(x'') + x_{12} \cdot x_{22}, x_{21} \cdot h_B(x'') \cdot x_{22}^2)$$

which is smooth.

§12

12.1 See end of book.

12.2
(a) We have seen that the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is given by

$$\tilde{\psi} \circ \tilde{\phi}^{-1}(x, a) = (\psi \circ \phi^{-1}(x), J(\psi \circ \phi^{-1}(x))(a)).$$

The Jacobian is then

$$J(x, a) = \begin{bmatrix} J(\psi \circ \phi^{-1})(x) & 0 \\ \star & J(\psi \circ \phi^{-1})(x) \end{bmatrix}$$

(b) The determinant becomes

$$\det J(\psi \circ \phi^{-1})(\phi(p)) \cdot \det J(\psi \circ \phi^{-1})(\phi(p)) = (\det[\frac{\partial y_i}{\partial x_j}](p))^2.$$

§13

13.1 See end of book.

13.2 See end of book.

13.4 We have that

$$(F^*h)^{-1}(R^*) = F^{-1}(\{h^{-1}(R^*)\}) \subset F^{-1}(\{\overline{h^{-1}(R^*)}\}) = F^{-1}(\mathrm{supp}(h)).$$

Then

$$\mathrm{supp}(F^*h) = \overline{(F^*h)^{-1}(R^*)} \subset \overline{F^{-1}(\mathrm{supp}(h))} = F^{-1}(\mathrm{supp}(h)),$$

where the last equality holds because $\mathrm{supp}(h)$ is closed and F is continuous.

13.6

(a) Fix a point $p \in N$. Then there is a neighbourhood V of $F(p) \in M$ and $\alpha_1, ..., \alpha_m$ such that $\mathrm{supp}(\rho_\alpha) \cap V \neq \emptyset \Rightarrow \alpha = \alpha_j$ for some $j = 1, ..., m$. Then $\mathrm{supp}(F^*\rho_\alpha) \cap F^{-1}(V) \neq \emptyset \Rightarrow \alpha = \alpha_j$ for some $j = 1, ..., m$.

(b) Since $\rho_\alpha \in C^\infty(M)$ it is clear that $F^*\rho_\alpha \in C^\infty(N)$. We have $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ and so $F^{-1}(\mathrm{supp}(\rho_\alpha)) \subset F^{-1}(U_\alpha)$. By **13.4** we have that $\mathrm{supp}(F^*\rho_\alpha) \subset F^{-1}(\rho_\alpha)$. Finally, for any $p \in N$ we have that

$$\sum_\alpha F^*\rho_\alpha(p) = \sum_\alpha \rho_\alpha(F(p)) = 1.$$

§14

14.1 See end of book.

14.2 Set $\rho(x,y) = \sum_{i=1}^n x_j^2 + y_j^2$. The tangent space at a point of a hypersurface in $\mathbb{R}^n$ is defined to be the orthogonal complement of a normal vector at that point. The gradient of ρ is normal to S^{2n-1}, i.e.,

$$\nabla\rho(x,y) = 2(x_1, ..., x_n, y_1, ..., y_n)$$

is normal at a point $(x,y) \in S^{2n-1}$. Since $(-y,x) \cdot (x,y) = 0$ it follows that the vector $(-y,x)$ is tangent to S^{2n-1} at (x,y). Writing the tangent field $X = (-y,x)$ as a vector field we have that $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Finally, X is even non-vanishing on $\mathbb{R}^{2n} \setminus \{0\}$.

14.9 Suppose $f \in C^\infty_{c_s(t_0)}$. Then

$$\frac{d}{dt}|_{t=t_0} f(c_s(t)) = \frac{d}{dt}|_{t=t_0} f(c(t-s))) = X_{c(t_0-s)}f = X_{c_s(t_0)}f.$$

14.10 See end of book.

14.12 See end of book.

§15

15.3 See end of book.

15.5 See end of book.

15.7 See end of book.

§16

16.5 See end of book.

16.6 For $g \in \mathbb{R}^n$ we have that $l_g x = g + x = x + g$ for all $x \in \mathbb{R}^n$. So $(l_g)_{*,0} = \text{id}$. So $(l_g)_{*,0} A = A$ for any $A \in T_0\mathbb{R}^n$ and every $g \in \mathbb{R}^n$, and so any left invariant vector field is constant.

16.7. We have seen that $\tilde{A}_g^m = gA_e^m$ (here the m means matrix-form). For $c(t) = e^{tA_e^m}$ we have seen that $c'(t)_{c(t)}^m = e^{tA_e^m} A_e^m = c(t)A_e^m = \tilde{A}_{c(t)}^m$, and it follows that c is an integral curve for $\tilde{A}$.

Next consider $\tilde{c}(t) = l_g c(t)$. Then $\tilde{c}(0) = g$. Morover

$$\tilde{c}'(t) = l_{g,*}c'(t) = l_{g,*}\tilde{A}_{c(t)} = l_{g,*}(l_{c(t)})_* A = (l_{gc(t)})_* A = \tilde{A}_{\tilde{c}(t)},$$

which means that $\tilde{c}$ is an integral curve through g.

§17

17.2 See end of book.

17.5 We check (i). We have that

$$F^*(\omega + \tau)X_p = (\omega + \tau)F_* X_p = \omega(F_* X_p) + \tau(F_* X_p) = F^*\omega X_p + F^*\tau X_p.$$

17.6 This was covered in class (see slides).

§18

18.2 Proof similar to that in 17.5

18.3 Covered in class (see slides).

18.8 See end of book.

§19

19.2 See end of book.

19.3 See end of book.

19.5 See end of book.

§21

21.3 Let $\mathcal{A}, \mathcal{B}, \mathcal{G}$ be oriented atlases, and consider the relation $\sim$ given in the problem. It is clear that $\sim$ is reflexive. Suppose $\mathcal{A} \sim \mathcal{B}$. Then

$$\det(J(\psi_\beta \circ \phi_\alpha^{-1})) = \frac{1}{\det(J(\phi_\alpha \circ \psi_\beta^{-1}))}$$

implies that $\mathcal{B} \sim \mathcal{A}$. So we have symmetry. Suppose further that $\mathcal{B} \sim \mathcal{G}$. Then

$$\det(J(\phi_\alpha \circ \sigma_\gamma^{-1})) = \det(J(\phi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta \circ \sigma_\gamma^{-1})) = \det(J(\phi_\alpha \circ \psi_\beta^{-1}) \cdot \det(J(\psi_\beta \circ \sigma_\gamma^{-1}))$$

which shows that $\mathcal{A} \sim \mathcal{G}$. So we have transitivity.

§22

22.4 See end of book.

22.6 See end of book.

§23

23.1 See end of book.

23.4 See end of book.

§24

§25

25.3 We show exactness at $H^k(\mathcal{B})$. Fix a class $[b_k]$. If $[b_k] \in \mathrm{Im}(i^*)$ it means there is $[a_k]$ such that $[ia_k] = [b_k]$ which means that $ia_k - b_k = db_{k-1}$. Then $j(ia_k - b_k) = d(jb_{k-1})$, and since $j \circ i = 0$ this means that $j(b_k) = d(-jb_{k-1})$, and so $[jb_k] = 0$. This shows that $\mathrm{Im}(i) \subset \mathrm{Ker}(j)$.

Suppose next that $j^*[b_k] = 0$. This means that $jb_k = dc_{k-1}$, and there is a b_{k-1} with $jb_{k-1} = c_{k-1}$. Then $j(db_{k-1}) = jb_k$ and so there is an a_{k-1} with $ia_k = db_{k-1} - b_k$. By injectivity of i at level $k+1$ we have that $da_k = 0$, and it follows that $[b_k] = i^*[a_k]$. This shows that $\mathrm{Ker}(j) \subset \mathrm{Im}(i)$.

§26

26.1 Let $\omega \in \Omega^k(M)$. If the restriction to both U and V is zero, it is clear that ω is zero, so i is injective. Let $(\omega, \tau) \in \Omega^k(U) \oplus \Omega^k(V)$. If (ω, τ) is in the image of i it means that $\omega = \tau$ on $U \cap V$ so it is in the kernel of j. If (ω, τ) is in the kernel of τ they are equal on $U \cap V$ and so they define a k-form on M whose image under i is (ω, τ).

26.2 See end of book.

§27

We consider the sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ for $n \geq 2$. We cover S^n by two open sets $U^n_\pm$ where

$$U^n_\pm = \{x = (x_1, ..., x_{n+1}) \in S^n : x_{n+1} \neq \pm 1\}.$$

We let $\varphi_\pm : U^n_\pm \to \mathbb{R}^n$ denote the stereographic projections. Then $\varphi_\pm$ are homeomorphisms between $U^n_\pm$ and $\mathbb{R}^n$ (where S^n is endoved with the subspace topology), and $\{(U_\pm, \varphi_\pm)\}$ gives S^n the structure of a smooth manifold.

(a) Prove that $U^n_+ \cap U^n_-$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$.

(b) Construct a homotopy equivalence between $\mathbb{R}^n \setminus \{0\}$ and S^{n-1}.

(c) Compute $H^k(S^n)$ for $k = 0, 1, 2, ...$ by induction on $n \geq 1$ (recall that we already computed $H^k(S^1)$ so you don't have to repeat that).