15th April, 2021

MAT4520 - Spring 2021

Mandatory assignment 1 of 1

Submission deadline

Thursday 22th April 2021, 14:30 in Canvas.

Instructions

The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number. Submission in LaTeXis preferred.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let V be a 3-dimensional vector space over $\mathbb{R}$ with a basis $\mathcal{B} = \{e_1, e_2, e_3\}$. We will consider the graded algebra $\oplus_{k=0}^{\infty} A_k(V)$ (here $A_0(V) = \mathbb{R}$, and $A_k(V) = 0$ for $k > 3$). Let $\{e_1^*, e_2^*, e_3^*\}$ the dual basis to $\mathcal{B}$, and recall that

$$\{e_1^* \wedge e_2^*, e_1^* \wedge e_3^*, e_2^* \wedge e_3^*\} \text{ and } \{e_1^* \wedge e_2^* \wedge e_3^*\}$$

are bases for $A_2(V)$ and $A_3(V)$ respectively.

We will now define a map $\star : \oplus_{k=0}^{3} A_k(V) \to \oplus_{k=0}^{3} A_k(V)$. Define $\star : A_1(V) \to A_2(V)$ by setting $\star e_1^* = e_2^* \wedge e_3^*, \star e_2^* = -e_1^* \wedge e_3^*$, $\star e_3^* = e_1^* \wedge e_2^*$, and extend by linearity. Define $\star : A_2(V) \to A_1(V)$ such that $\star^2 = \text{id}$. Define $\star : A_0(V) \to A_3(V)$ by setting $\star\lambda = \lambda e_1^* \wedge e_2^* \wedge e_3^*$ for $\lambda \in \mathbb{R}$, and define $\star : A_3(V) \to A_0(V)$ such that $\star^2 = \text{id}$. (This operator is called the *Hodge-$\star$* operator.) Finally, for each $k = 0, 1, 2, 3$, and $\alpha, \beta \in A_k(V)$, we define $\langle \alpha, \beta \rangle = \star(\alpha \wedge \star\beta)$.

(a) Show that for each k we have that $\langle \cdot, \cdot \rangle$ defines a symmetric bilinear form

$$\langle \cdot, \cdot \rangle : A_k(V) \times A_k(V) \to \mathbb{R}.$$

(b) Show that for each k we have that $\langle \cdot, \cdot \rangle$ is an inner product on $A_k(V)$, and find an orthonormal basis for each $A_k(V)$.

(c) For $\alpha, \beta \in A_1(V)$ we now define $[\alpha, \beta] = \star(\alpha \wedge \beta)$. Show that $[\cdot, \cdot]$ is an anticommutative bilinear form on $A_1(V)$ (it is in fact a Lie bracket).

(d) For $V = \mathbb{R}^3$ and $\mathcal{B}$ the standard basis on $\mathbb{R}^3$, express the relation between $[\cdot, \cdot]$ and the cross-product on $\mathbb{R}^3$.

Problem 2. Let M be a smooth manifold, and let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on M. Suppose that for any pair α, β with $U_{\beta\alpha} = U_\beta \cap U_\alpha$ nonempty, we have a smooth map $\phi_{\beta\alpha} : U_{\beta\alpha} \to GL(n, \mathbb{R})$, we assume that $\phi_{\alpha\alpha} = \text{id}$ for all α, and we assume that for all α, β, γ we have that

$$\phi_{\alpha\gamma}(x) \circ \phi_{\gamma\beta}(x) \circ \phi_{\beta\alpha}(x) = \text{id},$$

whenever $x \in U_{\gamma\beta\alpha}$, where $U_{\gamma\beta\alpha} = U_\gamma \cap U_\beta \cap U_\alpha$. On the disjoint union of the sets $U_\alpha \times \mathbb{R}^n$ we define a relation $\sim$ by declaring that $(x_\alpha, v_\alpha) \sim (y_\beta, w_\beta)$ if and only if $x_\alpha = y_\beta$ (both may be identified with points in M) and $\phi_{\beta\alpha}(x_\alpha)(v_\alpha) = w_\beta$.

(a) Show that $\sim$ is an equivalence relation. (Hint: show first that $\phi_{\alpha\beta} \circ \phi_{\beta\alpha} = \text{id}$ for all α, β.)

(b) Equip

$$E = (\sqcup_\alpha (U_\alpha \times \mathbb{R}^n)) / \sim$$

with the quotient topology, and define a natural projection $p : E \to M$. Show that for each $x \in M$ we have that $E_x = p^{-1}(x)$ has a natural structure of an n-dimensional real vector space.

(c) Show that E has the natural structure of a smooth manifold. (Hint: for each α, define a local chart by $[(x_\alpha, v_\alpha)] \mapsto (\phi_\alpha(x_\alpha), v_\alpha)$.)

(d) Such an object E is called an n-dimensional *vector bundle* over M. Show that TM is an n-dimensional vector bundle if M has dimension n.